Number Theory in the Medieval Islamic World

The Works of Ibn al-Haytham, Omar Khayyam, and Ibn al-Banna al-Marrakushi

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Winter 2023
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MATH377 - Jonathan Love

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Introduction

During the reign of al-Mamun (809-833), the Abbasid caliph is said to have had a dream in which Aristotle appeared. Having understood this as a calling for carrying the Greek tradition, the dream resulted in an Arab passion for translation. Arabic version of Ptolemy's *Almagest*, Euclid's *Elements*, and countlessly many other manuscripts were established. This initiated a time of mathematical flourishing in the medieval Islamic world, which built on previous Greek and Indian advancements.[1] The original title of this paper was *Number Theory in the Islamic Golden Age*, but the expression of "golden age" can be both inaccurate and misleading. While it may elegantly describe a period spanning from the 8th century to the 13th century, in which the Islamic world, stretching from China to the Iberian Peninsula, achieved spectacular economic, scientific, and cultural heights, it implies the existence of a "silver or bronze age". Hinting at the fact that this epoch of intellectual height is long lost, never to be seen again. Furthermore, it is important to note that the categorization as "golden age" began to be used in 19th century western orientalist literatures. This historical orientalism manifested itself through patronizing Western attitudes towards Middle Eastern, North African, and Asian societies and histories. According to Edward Said, these attitude essentialized

[1] Boyer, Carl B. , Merzbach, Uta C. 2011. "A History of Mathematics". Chapter 11: The Islamic Hegemony. John Wiley & Sons.

regional complexities to fabricate a fetishized view of the "Orient", at the service of Imperial dynamics.

This period of scientific, artistic, philosophical, and theological renewal is said to have started with the reign of the Abbasid caliph Harun al Rashid (786 -809), after the inauguration of the House of Wisdom (بيت الحكمة) in the late 8th century.[2] His marvelous rule is famously portrayed in *The Arabian Nights.* This institution was initially founded as a library for the Caliph's collections. Other sources indicate that the House of Wisdom was actually founded as a by an earlier caliph, Al Mansur, who reigned from 754 to 755 (some sources even associate the institution's foundation with caliph al-Mamun - 809-833 -). Nonetheless, what we are certain of is that this institution began to function as a major translation center. Caliphs and wealthy families employed translators, which used the athletic potential of the Arabic language in expressing subtle ideas and concepts. Similar to Latin during the Renaissance in Europe, Arabic would become a de-facto language of knowledge. Monumental Greek, Indian, Persian scientific works were translated into Arabic. This translation movement served as the base case for centuries of intellectual and cultural flourishing.

Mathematics was one of the major disciplines of focus during this age. Countless polymaths would become pioneers in the history of our beloved science: al-Khwarizmi, al-Kashi, Omar Khayyam, Thabit ibn Qurra, al-Karaji, Al-Biruni, Ibn al-Haytham, Ibn al-Banna' al-Marrakushi, and many others. Important progress was made in arithmetic, major innovations in trigonometry were established, and a unifying theory of Algebra was founded. Greek mathematics was based on geometry, but after developments in the Islamic world, rational numbers, irrational numbers, geometric magnitudes began to be treated as "algebraic objects", and the bridge between algebra and geometry began.[3] By combining and continuing Greek and Indian thought processes, mathematics was on a new trajectory of discovery and progress. Number theory, which will be the focus of this paper, did not occupy the same place it does today in medieval Islamic mathematics. Well before Gauss coined the field to be "the queen of mathematics", number theory was a subset of arithmetic, geometry, and algebra. Hence, a non-anachronistic perusal of number theory's history recommends a more unified study of mathematical branches as a path to comparative understandings of how different epochs interpreted concepts, which we could today associate with number theory, abstract algebra, or real analysis... The works and individuals mentioned below are in no way exhaustive in the history of medieval Islamic number theory. In that regard, our focus will be on a modest, but dense and rich, amount of work by Ibn al-Haytham, Omar Khayyam, and Ibn al-Banna' al-Marrakushi. These three mathematicians were specifically chosen for their illustration of the geographic and cultural diversity of the medieval Islamic world, representing its different regions and centers of knowledge.

On Hindu-Arabic Numerals

Among the faculty members at Baghdad's House of Wisdom was Muḥammad ibn Musa al-Khwarizmi (780 – 850), Persian polymath, often called the father of Algebra, was a pioneer in the history of mathematics. A textbook of his on arithmetic, *Algorithmo*

[2] Lyons, Jonathan. 2009. *The House of Wisdom. How the Arabs Transformed Western Civilization.* Bloomsbury Press.

[3] "Arabic Mathematics." Maths History. 2023. https://mathshistory.st-andrews.ac.uk/HistTopics/Arabic_mathematics/

de Numero Indorum (Concerning the Hindu Art of Reckoning), which exists only in Latin translation, the Arabic version having been lost, was probably based on translations works by Indian mathematician Brahmagupta. In it, al-Khwarizmi gave such a detailed account of the Hindu numerals that he is possibly responsible for their consequent widespread use to this day. Though he made no claim to originality in their use and application, when this number system was introduced to Europeans in the 10th century by Arabic speakers of Spain and North Africa, Latin translators began to attribute the numeration system to Al-Khwarizmi.[4]

(٠ ,١, ٢, ٣, ٤, ٥, ٦, ٧, ٨,٩) = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

The Hindus did not extend this system to represent parts of the unit by decimal fractions, and, since it was Islamic mathematics, who first did so, medieval Muslims were the first people to represent numbers in our modern sense. This is why our current base-10 system is, quite properly, referred to as the Hindu-Arabic numeral system.[5]

Ibn al-Haytham (965 – 1040) ابن الهيثم

Sometimes described as the world's first true scientist, Ibn al-Haytham was born in 965, in Abbasid Basra, modern day Iraq. Also known by his latinized name *Alhazen*, derived from al Hasan, his works were studied by Galileo, Kepler, Fermat, Snell, and Descartes. We know little of Ibn al Haytham's life in Basra. In his autobiography, he mentions that the conflicting views of different religious movements in the region led to him concluding that none of them represented the objective truth.[6] It seems that he did not devote himself to the study of mathematics or other academic fields at a young age. Rather, he was trained for a civil service job. Appointed as a minister for Basra, he grew dissatisfied with the job and decided to devote himself entirely to the study of science. The works of Aristotle were of paramount importance to him. Mathematics, physics, and other fields would become his life purpose.[7] At that time, the Fatimid dynasty (909 – 921) ruled over North Africa. Fatimid caliph al-Hakim (996-1021) had a cruel, murderous, and eccentric reputation.[8] Historical accounts of his rule are diverse and often controversial for scholars. Nonetheless, he was known to be a patron of science, employing many brilliant scholars from around the world (For more information, see *A Short History of the Fatimid Khalifate*, by De Lacy O'Leary). Al-Hakim also founded Cairo's *House of Knowledge*, Dar al-'Ilm (دار العلم), a grand library, then university, which attracted numerous mathematicians, astronomers, philologists, jurists, physicians,

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[4] Boyer, Carl B. , Merzbach, Uta C. 2011. "A History of Mathematics". Chapter 11: The Islamic Hegemony. John Wiley & Sons.

[5] Berggren, J.L. 2016. "Episodes in the Mathematics of Medieval Islam." *SpringerLink*. https://doi.org/10.1007-978-1-4939-3780-6. 32.

[6] "Ibn Al-Haytham - Biography." 2013. Maths History. 2013. https://mathshistory.st-andrews.ac.uk/Biographies/Al-Haytham/.

[7] Ibid.

[8] Nkrumah, Gamal. 2019. "Al-Ahram Weekly | Culture | the Crazed Caliph." Archive.org. 2019.https://web.archive.org/web/20130327053334/http://weekly.ahram.org.eg/2009/976/cu3.htm.

grammarians, and other scholars alike. The university was welcoming to all and remained largely unaffected by political turmoil. After devising a method to control water flows down the Nile, al-Hakim invited Ibn al Haytham to Egypt to lead an engineering team and implement his ideas. He refused after a field visit which made him realize the project was unfeasible. Furious, the Caliph placed under protective custody for 10 years. After the death of al-Hakim, Ibn al Haytham would resume his writing in teaching in Egypt, where he resided near the Al-Azhar Mosque for the rest of his life. Often considered his most important work, the Kitab al-Manazir (كتاب المناظر), or *Book of Optics*, was the first modern description of physical light rays and their reception by the eye.[9] His theories replaced the confused debate that had occurred among classical Greek thinkers. The book is noted for its use of the scientific method, wherein Ibn al Haytham conducted a series of experiments on the rectilinear properties of light using a dark room with slits in an intermediate wall. His work on optics is immensely rich and important, so much that he is often referred to as the father of modern optics. Given the large corpus of his work, an attempt to summarize it would be dishonorable to his genius. Here, we discuss his mathematical works, specifically in number theory. His most notable contribution in the discipline is his work on solving congruences and perfect numbers.

Definition: A natural number $n > 1$ is a prime number if, and only if, the product of all natural numbers less than n is one less than a multiple of n.

$$n \text{ is prime if, and only if, } (n-1)! \equiv -1 \pmod{n}$$

This statement is known as Wilson's theorem. Edward Waring stated it in 1770, but neither he, nor his student John Wilson, could prove it. Lagrange gave the first proof in 1771. Ibn al Haytham introduced this theorem, in his *Opuscula*, while solving linear congruences, as a proposition that states an exclusive property of prime numbers. We will follow Ibn al Haytham's order of solution to observe how he set up the theorem. In his *Opuscula*, he proposes to solve the system:

$$(1)\quad \begin{cases} x \equiv 1\ (mod\ m_i) \\ x \equiv 0\ (mod\ p) \end{cases}$$

With p a prime number and $1 < m_i \leq p$ - 1. This is a special case of the Chinese remainder theorem. First establishing that the system presents an infinitely many solutions, Ibn al Haytham then suggests two methods for finding these solutions. The first method is based on Wilson's theorem:

> If p is any prime number, then the sum $2\times 3 \times \ldots \times (p-1) + 1$ is divisible by p; and if we divide it by any natural number 2, 3, …, $(p-1)$, the remainder will always be 1. This gives us a solution to the system (1):
>
> $$(2)\quad x = (p-1)! + 1$$

[9] Smith, John D. "The Remarkable Ibn Al-Haytham." *The Mathematical Gazette* 76, no. 475 (1992): 189–98.

Ibn al Haytham then proposes a second method for solving the congruence:[10]

(a) If m is the lcm of m_i; then gcd (p, m) = 1
(b) If x_0 is a solution to the first equation of (1).

Then the general solution is $x = x_0 + \lambda m$ such that λ $is\ an\ arbitrary\ integer$.

(c) if r is such that $m \equiv r\ (mod\ p)$ then gcd (r, p) = 1

$$(3)\ \begin{cases} x \equiv 1\ (mod\ m) \\ x \equiv 0\ (mod\ p) \end{cases}$$

We now find an s such that,

$$(4)\ \begin{cases} \mathrm{s} - 1 \equiv 1\ (mod\ r) \\ x \equiv 0\ (mod\ p) \end{cases}$$

s = p + kp satisfies (4). Hence, by taking the smallest k such that s satisfies the first equation in the system, we get:

$$(5)\ (p-1) + kp \equiv 0\ (mod\ r)$$

Ibn al Haytham states that this is only possible if gcd (p, r) = 1, i.e., there exist integers k an h such that (6): (k + 1) p - hr = 1.
Let k_0 and h_0 be the smallest integers satisfying (6). We thus get:

$$\mathrm{s} = p + k_0 p \quad \text{or} \quad \mathrm{s} = 1 + h_0 p$$

Hence, $h_0 = \frac{s-1}{r}$. Consider $\frac{m\ (s-1)}{r} + 1$, it verifies the first equation of (3).

A smaller solution would be $x = mh_0 + 1$, with the general solution being:

$$x = m(h_0 + \mathrm{n}p) + 1 \equiv (mh_0 + 1)(mod\ p)$$

While this section mostly uses the works of mathematical historian Roshdi Rashed[11], the following is an original example worked out by the author. Consider the following system:

$$\begin{cases} x \equiv 1\ (mod\ 2) \\ x \equiv 0\ (mod\ 3) \end{cases}$$

Using Wilson's theorem, we find that (3-1)! + 1 = 3

$$3 \equiv 1\ (mod\ 2)\ and\ 3 \equiv 0\ (mod\ 3) \text{ thus, 25 is a solution.}$$

[10] Rashed, Roshdi. 2013. "The Development of Arabic Mathematics: Between Arithmetic and Algebra." *SpringerLink*. https://doi.org/10.1007-978-94-017-3274-1. 238.
[11] Ibid.

Consider his second method generating infinitely many solutions, let $1 < m_i \leq 3 - 1$. Therefore lcm $m_i = \text{lcm}\ (2{,}1) = 2 := m\ \equiv 2\ (mod\ 5). Thus\ r := 2.$
Let us now find s such that

$$\begin{cases} s - 1\ \equiv\ 1\ (mod\ 2) \\ x\ \equiv\ 0\ (mod\ 3) \end{cases}$$

$3 + 3\text{k}$ satisfies this system of equations, we thus take the smallest integer k satisfying the first equation of the system we get $(3-1) +\ k3 \equiv\ \ 0\ (mod\ 2)$

$$2 +\ 3k \equiv\ \ 0\ (mod\ 2)\ , 8 \equiv 0\ (mod\ 2) \text{ for k} = 2.$$

$$\text{Gcd } (3, 2) = 1 \text{ thus, } (\text{k} + 1)\ 3 - 2\text{h} = 1$$
$$3\ \times 3 - 2 \times h = 1\ \rightarrow h = 4$$

Thus let $k_0 = 2\ and\ h_0 = \frac{s-1}{r}\ ,\ \frac{3-1}{2} = 1.$

The general solution is thus, $x = m(h_0 + \text{n}p) + 1 = 2(1 + 3\text{n}) + 1 \equiv 0\ (mod\ 3)$
$and\ x = m(h_0 + \text{n}p) + 1 = 2(1 + 3\text{n}) + 1 \equiv 1(mod\ 2).$

Of the two methods presented, the second is sufficient as it provides all solutions to the system of congruences. But ibn al Haytham insisted on presenting the first approach. This probably means that arriving at Wilson's theorem was his aim, based on uses of the Chinese remainder theorem, since his successors only retained the second approach to solving systems of congruences.[12] Some scholars propose that these confusions in progression of intellectual history are due to our ignorance of the exact knowledge in number theory of the time.[13] The large number of lost texts from that period supports this interpretation. Ibn al Haytham seems to have had the tools to prove Wilson's theorem, especially given the constantly rigorous scientific requirements he established in his works. Nonetheless, claiming he proved Wilson's theorem remains mere conjecture.[14]
The father of optics also studied perfect numbers: a positive integer that is equal to the sum of its divisors excluding the number itself. For example:

$$6 = 1 + 2 + 3, \text{ making 6 a perfect number.}$$

In his *Analysis and Synthesis* (مقالة في التحليل والتركيب), Ibn al Haytham was the first to conjecture that every even perfect number was of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime. He was not entirely successful in this attempt to classify the set of perfect numbers; however, it was later proved by Euler in the 18th century and is now known as the Euclid-Euler theorem.[15]

[12] Rashed, Roshdi. 2013. "The Development of Arabic Mathematics: Between Arithmetic and Algebra." *SpringerLink*. https://doi.org/10.1007-978-94-017-3274-1. 242.
[13] Ibid.
[14] Ibid.
[15] Rashed, Roshdi. « L'analyse et la synthèse selon Ibn al-Haytham », Roshdi Rashed éd., *Mathématiques et Philosophie. De l'Antiquité à l'Âge classique.* CNRS Éditions, 1991, pp. 131-162.

Omar Khayyam (1048 – 1131) عمر خيّام

About a century after Ibn al Haytham came another great polymath, known equally for his poetry, philosophy, and mathematics: Omar Khayyam. In his *Ruba'iyyat* (quatrains), wrote:

Since neither truth nor certitude is at hand
Do not waste your life in doubt for a fairyland
O let us not refuse the goblet of wine
For, sober or drunken, in ignorance we stand.[16]

Western discussions of Khayyam often involve erotic descriptions of a stereotypical and cynical hedonistic, agnostic, philosopher, but this is pure distortion. Described as a spiritual pragmatist, his philosophical and mathematical works deal with questions of uncertainty, tribulations of daily life, the human existence, and its relation to the universal, often through rational thought. With a distaste for religious orthodoxy, a lust for the beauties of life, Omar Khayyam was interested by the unsolvable question of truth. His poetry was discussed by the likes of Mark Twain and Ezra Pound.[17]

Of Khorasani Persian ancestry, Khayyam was born in Nishapur, in 1048, where he spent his childhood. The region of southwestern Asia was under the rule of the Sejluq Turks at the time, whose military rule would shape much of Omar Khayyam's life. His intellect was noticed early on, his tutors sent him to study under the greatest teacher of the Khorasan region: Imam Muwaffaq Nishaburi, with whom he would develop a beautiful friendship over the years. He would also be taught by Abu al-Hasan Bahmaynar ibn al-Marzuban, a Zoroastrian mathematician and student of Avicenna. Khayyam referred to Avicenna as his master, and some say that he was his student, though historians consider this to be very unlikely due to time discrepancies. Around 1068, Khayyam traveled to the province of Bukhara, in modern day Uzbekistan, known for its grand library of the Ark, which he perused. Around 1070, after moving to the great city of Samarkand, Khayyam started to work on his *Treatise on Demonstration of Problems of Algebra*, under the patronage of Abu Tahrir, a prominent jurist of the time. Before Khayyam's time, Toghril Beg, founder of the Sejluq dynasty had made Esfahan capital of his domains and his grandson Malik-Shah ruled the city beginning in 1073, along his vizier Nizam al Mulk, a friend of Khayyam. He would invite him to set up an observatory and lead a team of astronomers. While in Isfahan, Khayyam found himself around opulent palaces, and a luxurious life, to which he was unaccustomed. Coming from a modest background, Khayyam was known for his solitary attitude, and his noble humility. His Vizier friend, Nizam al Mulk, noticed this discomfort and catered to it. The only riches Khayyam needed were the rich libraries of the Sejluq royal courts, which contained treasures such as the works of Euclid. This was a period of peace and protection in Khayyam's life, for 18 years, he would produce astounding work in many fields. However, in 1092, Malik-Shah died, and his Nizam al Mulk was murdered by the Ismaili order of the Assassins. Khayyam then decided to go to Mecca to preform

[16] Aminrazavi, Mehdi and Glen Van Brummelen, "Umar Khayyam", *The Stanford Encyclopedia of Philosophy* (Winter 2022 Edition).
[17] Schenker, Daniel. "Fugitive Articulation: An Introduction to 'The Rubáiyát of Omar Khayyam.'" *Victorian Poetry* 19, no. 1 (1981).

pilgrimage.[18] The motives for this journey remain unclear, some tell us that it was a way for him to prove his faith to exonerate himself of allegations of heresy. At his return, the new sultan Sanjar invited him to stay at his court, Khayyam accepted and moved to Marv. Declining in health, Khayyam demanded the Sultan if he could return to his native city of Nishapur. The rest of his life would be dedicated to scholarly and scientific writings, which were not large in number, but ground-breaking, dense, and exceptional. From his *Ruba'iyyat*, we read and understand this devotion to knowledge:

> Of knowledge naught remained I did not know,
> Of secrets, scarcely any, high or low.
> All day and night for three score and twelve years,
> I pondered, just to learn that naught I know.[19]

According to his son in law, Imam Muhammad Baghdadi, on his final day, Khayyam refused to eat or drink until he performed his night prayer. He prostrated by putting his forehead on the ground and said, "O Lord, I know you as much as it is possible for me, forgive me for my knowledge of you is my way of reaching you" and then died.[20]

His works in mathematics are: *On the elaboration of the problems concerning the books of Euclid,* in which he provides proofs for the incompleteness of certain principles in Euclidean geometry, and calls for philosophical investigations of these principles; *On the division of a quadrant of a circle* which is a work on the historical achievements of previous mathematicians; and *Treatise on Demonstration of Problems of Algebra*, where he highlighted basic algebraic principles and, with regards to number theory, provided various solutions of the cubic equation, which we will presently entertain.

He was the first to classify equations modern terms, i.e., referring to the degree of an equation.[21] The first set of equations (1) are those with two terms, which he called "simple equations", the second set (2) are what he called "compound equations", which are divided into trinomial quadratic equations, trinomial cubic equations, and tetranomial equations in which the sum of three terms is equal to the fourth term:

(1) $a = x\ ;\ a = x^2\ ;\ a = x^3\ ; ba = x\ ;\ ba = x^2\ ;\ va = x^3.$

(2) $Trinomial\ quadratic\ equations\colon\ x^2 + bx = a\ ;\ x^2 + a = bx\ ;\ x^2 = bx + a$

$Trinomial\ cubic\ equations\colon x^3 + bx = a\ ;\ x^3 + a = bx\ ; x^3 = bx + a\ ;$

$x^3 + cx^2 = a\ ;\ x^3 + a = cx^2\ ;\ cx^2 + a = x^3$

$Tetranomial\ equations\ of\ the\ form\colon x^3 + cx^2 + bx = a\ or\ x^3 + cx^2 = bx + a$

[18] Aminrazavi, Mehdi. 2019. "Chapter 1: Khayyam's Life and Works". *The Wine of Wisdom: The Life, Poetry and Philosophy of Omar Khayyam.*
[19] *Rubā'iyyāt*, Sa'idī 1991, p. 125
[20] Aminrazavi, Mehdi. 2019. "Chapter 1: Khayyam's Life and Works". *The Wine of Wisdom: The Life, Poetry and Philosophy of Omar Khayyam.*
[21] Mardia, K. V. 2004. "Omar Khayyam, René Descartes and Solutions to Algebraic Equations." *History of the Mathematical Sciences*, 135–48. https://doi.org/10.1007/978-93-86279-16-3_11.

Khayyam gave a history of which equations were solved already and gave methods of solutions according to respective categories. Although algebraically, some of these equations are identical, their geometric constructions differ. Notice that none of the coefficients are negative as this was not yet prevalent at the time due to the geometric interpretation to these equations which was thought to be necessary at the time (how could you have a negative square?).[22] We will now examine his method for finding solutions. The following theorem is to be proved

> **Theorem:** The four points of intersection of two parabolas whose axes are perpendicular lie on a circle.

Without loss of generality, we choose the two following parabolas.

$$y^2 = 4p(x-a)\ ;\ x^2 = 4q(y-b)$$

It is clear that the axes of these parabolas are perpendicular to one another. Let us add these equations, we get:

$$x^2 + y^2 - 4px - 4qy + 4ap + 4bq = 0$$

which is a circle of center (2p , 2q). This proves the theorem.
Any third-degree equation can be written in the form:

$$(2.1)\ x^3 + lx^2 + mx + n = 0$$

If we discuss the solution of a fourth-degree equation such as:

$$(2.2)\ z^4 + az^3 + bz^2 + zc + d = 0$$

Then (2.1) is a special case of (2.2), henceforth, we consider the following equation by omitting the trivial solution x = 0, to get the roots of (2.1).

$$(2.3)\ x^4 + lx^3 + mx^2 + nx = 0$$

Let us change of variable for (2.2) by setting $z = x - \left(\frac{a}{4}\right)$. We obtain:

$$(2.4)\ x^4 + Ax^3 + Bx + C = 0$$

If we choose $y = x^2$, then getting the solutions of (2.4) is the same as solving the system:

$$(2.5)\ \begin{cases} x^2 = y \\ y^2 + Ay + Bx + C = 0 \end{cases}$$

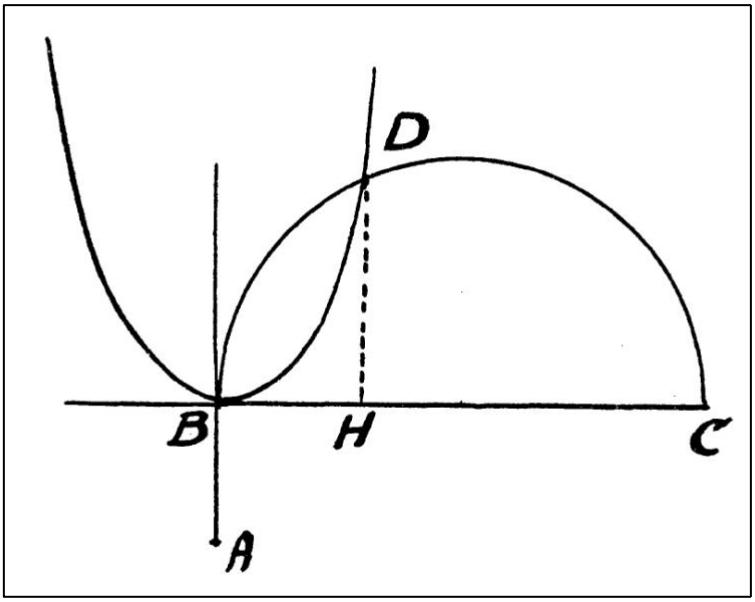
We remark that the equations of (2.5) are the equations of two parabolas which are perpendicular to one another. The solution of (2.5) is obtained below:

[22] Ibid.

$$(2.6)\ \begin{cases} x^2 = y \\ x^2 + y^2 + (A-1)y + Bx + C = 0 \end{cases}$$

$x^2 = y$ is a parabola which can be drawn. Similarly, a circle of center (A-1/2 , B/2) and of radius $\frac{(A^2+B^2-4AC-2A+1)^{\frac{1}{2}}}{2}$ can be superimposed on the parabola. Thus, we read the roots on the $x - axis$.

For example, the equation (3.1) $x^3 + Bx = C$ is solved by Khayyam in the following way. He rewrites (3.1) $x^3 + b^2x = b^2c$. Let AB = b and BC = c in the figure below. The parabola $x^2 = by$ and the half circle with diameter BC intersect at a point we call D. The rest of the argument uses geometry of numbers to solve the cubic.[23]



[24]

$x = DH\ is\ a\ root\ of\ (3.1)$

Proof:

$BH^2 = (AB)(DH)$. Thus, $\frac{AB}{BH} = \frac{BH}{DH}$ or $\frac{b}{x} = \frac{x}{DH}$

But in the circle, $\frac{BH}{HD} = \frac{HD}{HC}$. Therefore, $\frac{AB}{BH} = \frac{DH}{HC}$, or $\frac{b}{x} = \frac{DH}{c-x}$.

Finding DH, we thus find that $x^3 + b^2x = b^2c$ has one real root, which always exists and is obtained by the intersection between the circle and the parabola.[25]

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[23] Amir-Moéz, A R. 2022. "Khayyam's Solution of Cubic Equations." Mathematics Magazine. 2022.
[24] Ibid.
[25] Ibid.

Ibn al-Banna al-Marrakushi (1256 – 1321) ابن البناء المراكشي

Born in Marrakesh in 1256, Ibn al-Banna al-Marrakushi was a mathematician and astronomer, we know very little of his life, though he studied a variety of subjects, under at least 17 masters. His formative years were spent in Marrakech, but a portion of his adult life was also spent in Fez, which was at the time the capital of the Marinid Sultanate that ruled over what is now Morocco, other parts of North Africa, and southern Spain. The city was a center of culture and knowledge as it hosted of the most important institutions of education of the time, the University of al-Qarawiyyin where Ibn al Banna taught. The school's alumni include the likes of Ibn Arabi, Ibn Khaldun, Averroes, Leo Africanus, Maimonides, Pope Sylvester II. His catalog of works comprises of over 70 titles, around half of which are dedicated to mathematics and astronomy, the remaining part pertains to Quranic studies, theology (*uṣūl al-dīn*), logic, law (*fiqh*), rhetoric, prosody, Sufism, the division of inheritances (*farāʾiḍ*), weights and measures, measurement of surfaces (*misāḥa*), talismanic magic, and medicine.[26] His most notable mathematical works are his *Summary of Arithmetical Operations* (Talkhīṣ ʿamal al-ḥisāb), where he deals with fractions, sums of squares, and his *Lifting the Veil from Faces of the Workings of Calculations* (Rafʿ al-Ḥijāb 'an Wujuh A'mal al-Hisab) which covers square roots and the theory of continued fractions.[27]

In his *Summary of Arithmetical Operations*, Ibn al-Banna', like Diophantus, deals with this finding rational solutions x, y, z to $x^2 + y^2 = z^2$, and states the following properties:

(1) If a and b are any two numbers such that $\frac{a}{b} = \frac{3}{4}$. then $a^2 + b^2$ is a square of a rational number.

(2) If a is a square it may be expressed as the sum of two squares. His abbreviated demonstration of this is: "This is because there exist two squares whose square is a square. The given square is then decomposed according to their ratio."

(3) If a is not a square, then if there exist whole numbers x and y satisfying $a = x^2 + y^2$ there also exists another, different, pair of numbers, w, and z, such that $a = w^2 + z^2$.

(4) He concludes with the following procedure for deciding whether a whole number can be expressed as the sum of two squares:[28]

> You may know whether it has two square parts by subtracting from it the first of the natural squares, i.e., 'one.' And if the difference has a [whole number] root [then you have expressed it as the sum of two squares. But if not, one subtracts the second square, which is 'four' and one examines the remainder and one proceeds step-by-step in this fashion. If it is one of those numbers that cannot be expressed as the sum of two squares this will become evident

[26] "Ibn Al-Banna." 2023. Mcgill.ca. 2023. https://islamsci.mcgill.ca/RASI/BEA/Ibn_al-Banna%27_BEA.htm.

[27] "Ibn Al-Banna - Biography." 2023. Maths History. 2023. https://mathshistory.st-andrews.ac.uk/Biographies/Al-Banna/.

[28] Berggren, J L. 2014. "Episodes in the Mathematics of Medieval Islam." *SpringerLink*. 224.

with whole numbers. For if it cannot be decomposed into two whole number squares neither can it be decomposed into squares of fractions. Keep this in mind.[29]

In property (3), he specifies the necessity of natural numbers, otherwise, the use of the word number implies that the number is rational. From a modern number theoretic point of view, one may interpret his work on sums of squares as follows:

(1) The first property tells us to generate as many pairs as we want whose squares sum to a rational square, he found these pairs by using the Pythagorean theorem, i.e., a right triangle with rational sides p, and q. Let p, q, r be rational numbers such that $p^2 + q^2 = r^2$. If $\frac{p}{q} = \frac{m}{n}$, where m and n are rational, then $\left(\frac{p}{q}\right)^2 + 1 = \left(\frac{m}{n}\right)^2 + 1$. We get, $\frac{n^2r^2}{q^2} = m^2 + n^2$, i.e. $m^2 + n^2$ is also the square of a rational.

(2) As a motivating example, since 25=$5^2 = 3^2 + 4^2$, if one wishes to express 36 as a sum of two rational squares, it suffices to multiply 3^2 *and* 4^2 *by* $\frac{36}{25}$.

(3) If one finds a sum of squares for a rational number which is not a square, then exists other rational expressions for this sum of squares.

(4) This property gives an inductive procedure to determine whether a natural number is a sum of two squares. To our knowledge, he was unable to prove this result, it remains unclear how he was able to convince himself of this result. It was Fermat who proved for any natural n > 1, n is a sum of two squares if, and only if, all primes congruent to 3 (mod 4) in its prime factorization appear to an even power.[30]

An application of his properties follows: $10 = 1^2 + 3^2$, hence by property (3), there exists another expression for 10 as a sum of two squares. Multiple 10 by a square, say 25, we get 250 = 25 + 225.
Property (4) yields another decomposition of 250 as a sum of two squares, $250 = 9^2 + 13^2$, dividing again by 25, we get:

$$10 = \left(\frac{9}{5}\right)^2 + \left(\frac{13}{5}\right)^2.$$

Ibn al-Banna' assures that if this does not work for a square multiplier, in this case 25, then there will always be one for which it works.[31]

Although his work in number theory does not compare to that of Fermat, Euler, Gauss, or Lagrange, he did some original work in combinatorics, and proved interesting results. His manuscripts were studied until the early 20th century[32] and played an important role in continuing and promoting our mathematical tradition.

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[29] Ibid.
[30] Berggren, J.L. 2016. "Episodes in the Mathematics of Medieval Islam." *SpringerLink*. https://doi.org/10.1007-978-1-4939-3780-6. Pp 224.
[31] Ibid.
[32] Ibn Al-Banna." 2023. Mcgill.ca. 2023. https://islamsci.mcgill.ca/RASI/BEA/Ibn_al-Banna%27_BEA.htm.

Conclusion

This concludes my modest exposition of the number theory of the medieval Islamic world. Many mathematicians and their theorems have been omitted. However, it is my hope that the reader was able to taste a glimpse of this, frequently forgotten, grand tradition and time in the history of science.

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