Computing Zeroes of the Riemann Zeta Function

Zack Horton

April 13, 2023

Abstract

The aim of this project is to give a brief overview of the Riemann zeta function and to show one method used to calculate non-trivial zeros of the function on the critical line at $\frac{1}{2}$. Bernhard Riemann hypothesized that all non-trivial zeros lie on this critical line. If true, this has major implications about the distribution of prime numbers. The methods discussed in this paper give numerical evidence, but not proof, to suggest the truth of this hypothesis.

1 Zeta Function Background

1.1 Zeta Function Definitions

The Riemann Zeta function was originally defined by Dirichlet as the infinite series [4]:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

By the p-series test, we see this is convergent for s > 1, $s \in \mathbb{R}$. Euler was able to find another way to define it [8]. He noted that:

$$\zeta(s) = \prod_{p} \frac{1}{1 - \frac{1}{p^s}} \tag{2}$$

Where p ranges over all primes numbers. Bernhard Riemann used this as his point of departure for his paper "On Primes Less than a Given Magnitude" [8]. He noted that both definitions were convergent for $s \in \mathbb{C}$ such that Re(s) > 1. He went on to prove that it can be extended to $\mathbb{C} \setminus \{1\}$. An adaptation of his proof, with some additional details provided by Jekel[4], is given below.

Theorem 1. *$\zeta(s)$ extends analytically to $\mathbb{C} \setminus \{1\}$*

One first considers the equation:

$$\int_0^{\infty} e^{-nz} z^{s-1} dz = \frac{\Pi(s-1)}{n^s} \tag{3}$$

Where in this case Π is the pi-function defined as: $\Pi(s) = \int_0^{\infty} e^{-z} z^s dz$. Summing over n in (3) and using the geometric series formula in the integral gives (where the exchange of summation and integration is justified by convergence at endpoints):

$$\Pi(s-1) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} \frac{e^{-z} z^{s-1}}{1 - e^{-z}} dz = \int_0^{\infty} \frac{z^{s-1}}{e^z - 1} dz$$

Next to consider is the integral:

$$\int_\gamma \frac{(-z)^{s-1}dz}{e^z-1} = \int_\gamma \frac{e^{s(log(z)-\pi i}dz}{e^z-1} \tag{4}$$

Where γ is a closed, positively oriented curve such that γ encompasses zero but no other discontinuity point of the integrand. Let the section of the curve surrounding 0 be a circle of radius $0 < \delta < 1$. Then the absolute value of the parameterized integral around this portion is:

$$\left| \int_0^{2\pi} \frac{e^{s(log(\delta)-i\pi+i\theta)}}{e^{\delta e^{i\theta}}-1} id\theta \right| \leq \int_0^{2\pi} \left| \frac{e^{s(log(\delta)-i\pi+i\theta)}}{e^{\delta e^{i\theta}}-1} i \right| d\theta \leq \int_0^{2\pi} \frac{\delta}{e^\delta - 1}\delta^{Re(s)-1}e^{\pi Im(s)}d\theta$$

We see that when $Re(s) > 1$, taking $\delta \to 0$ gives that the integral over this part of the curve is 0. So we see that (4) becomes:

$$\lim_{\delta\to 0^+}\left(\int_\delta^\infty \frac{e^{s(log(z)-\pi i}dz}{e^z-1} + \int_\infty^\delta \frac{e^{s(log(z)-\pi i}dz}{e^z-1}\right) = (e^{i\pi s} - e^{-i\pi s})\int_0^\infty \frac{z^{s-1}}{e^z-1}dz$$

But from above we see this is exactly $\Pi(s-1)\sum_{n=1}^{\infty}\frac{1}{n^s}$. Using this, as well as some manipulations of the Π function, we can state the integral definition of ζ (Jekel, 2013):

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_\gamma \frac{(-z)^{s-1}dz}{e^z-1} \tag{5}$$

The integral converges on compact subsets of $\mathbb{C}$ because the growth of the numerator is outpaced by the growth of the denominator for any s. With this, we see that the integral is analytic. Therefore, the only possible poles of $\zeta(s)$ on $\mathbb{C}$ are where $\Pi(-s)$ has poles, which are at positive integers. However, since (5) agrees with (1) for $Re(s) > 1$ and converges for these values, we conclude that the only pole of $\zeta(s)$ in $\mathbb{C}$ is 1 [4]. □

Riemann also notes that this shows that negative even integers are zeros [8]. These have become known as the trivial zeros.

1.2 Motivation for Studying

Once defined for the complex plane, Riemann went on to define an analytic functional equation:

$$\xi(s) = \Pi(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) \tag{6}$$

for which $\xi(s) = \xi(1-s)$. I will merely reference Riemann's proof and use the result[8]. The zeros of ξ correspond to zeros of ζ since Π has no zeros on $\mathbb{C}$ (which follows from the proof that Γ has no zeros, found in section 2.2 of this paper). (2) shows that $\zeta(s)$ has no zeros for $Re(s) > 1$. This implies that ξ has no zeros in this region either. However, since $\xi(s) = \xi(1-s)$, we see ξ has no zeros for $Re(s) < 0$ either. Therefore, zeros of ξ and thus ζ are contained in the region of the complex plane between 0 and 1. This is commonly known as the critical strip. Riemann set $s = 1/2 + ti$ and conjectured that $\zeta(t)$ had only real roots. "And it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation"[8]. This is the translated statement of the Riemann hypothesis (RH). Many consider this to be one of the most important unsolved problems in modern mathematics.

Without going into too many details, Riemann goes on to define a function $f(x)$:

$$f(x) = Li(x) - \sum_{\alpha}(Li(x^{\frac{1}{2}+\alpha i}) + Li(x^{\frac{1}{2}-\alpha i})) + \int_{x}^{\infty} \frac{1}{x^2-1}\frac{dx}{xlog(x)} + log(\xi(0)) \tag{7}$$

Where α ranges over roots of $\xi(\alpha) = 0$ with $Re(\alpha) > 0$. The derivative of this function "gives an approximating expression for the density of the prime number + half the density of the squares of the prime numbers + a third of the density of the cubes of the prime numbers etc. at the magnitude x" [8].

It is in this way that Riemann's work, and the zeta function, are related to number theory. With this we begin to see the motivation for studying ζ and ξ. His famous hypothesis is related to the prime number theorem:

Theorem 2. *Prime Number Theorem*

$$\pi(x) \sim \frac{x}{log(x)} \tag{8}$$

Where $\pi(x)$ is the prime counting function defined as: $\pi(x) = \sum_{p<x} 1$. The prime number theorem also gives a bound for the difference of these functions, using the bounds of the region of non-trivial zeros [7]. If the Riemann hypothesis is true, it would give a tight bound on the error term in this theorem. Assuming the hypothesis:

$$\pi(x) = Li(x) + O(\sqrt{x}log(x)) \tag{9}$$

[5]

The proof of this relies on the fact that, for non-trivial roots ρ of $\zeta(s)$, we have $|x^\rho| = x^{1/2}$. In particular, that all non-trivial roots ρ have real part 1/2. I will omit the proof and refer the reader to chapter 4 of Jin's paper[5]. This error term cannot be improved by much due to known oscillations found by Littlewood [1].

This brings us to the reason for calculating zeros of the zeta function. Without proof, we can only gather numerical evidence that points to the validity of the Riemann hypothesis, and thus the validity of this tight approximation for $\pi(x)$. This in turn is of interest due to the importance of primes in various areas of mathematics and computer science.

2 Computation of Zeros

2.1 History

Mathematicians have been trying to find non-trivial zeros of $\zeta(s)$ for over a century. With the evolution of computing technology, we can trace the increasing success of this numerical verification. In addition, the algorithms designed to compute these zeros have improved over the years. The table below lists some of the mathematicians who worked on this project and the number of zeros they were able to calculate. A "history of the RH verification on the first n zeros" [3].

Year	n	Author
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1,041	E. C. Titchmarsh
1953	1,104	A. M. Turing
1956	15,000	D. H. Lehmer
1956	25,000	D. H. Lehmer
1958	35,337	N. A. Meller
1966	250,000	R. S. Lehman
1968	3,502,500	J. B. Rosser, J. M. Yohe, L. Schoenfeld
1977	40,000,000	R. P. Brent
1979	81,000,001	R. P. Brent
1982	200,000,001	R. P. Brent, J. van de Lune, H. J. J. te Riele, D. T. Winter
1983	300,000,001	J. van de Lune, H. J. J. te Riele
1986	1,500,000,001	J. van de Lune, H. J. J. te Riele, D. T. Winter
2001	10,000,000,000	J. van de Lune
2003	250,000,000,000	S. Wedeniwski

This table is from the 2004 paper by X. Gourdon in which he explains his method for calculating the first 10^{13} zeros. He makes sure to note that all of these calculations have agreed with Riemann's hypothesis; that the calculations have verified it to some number of zeros[3]. While this is no rigorous proof, it does have implications in of itself. For example, one can now say for that the tight error bound that results from the RH for $\pi(x)$ is valid up to a relatively large magnitude of x.

One caveat is that these numerical methods leave room for some errors. Whether it be human error, computer bug or a result of imprecision, the lack of rigorous proof leaves room for inaccuracies. However, with the advancements in computing one might expect to see similar numerical verification for other conjectures.

2.2 The Hardy Function

We now introduce the Hardy Function:

$$Z(t) = \pi^{-it/2} \frac{\Gamma(1/4 + it/2)}{|\Gamma(1/4 + it/2)|} \zeta(1/2 + it) \tag{10}$$

This is a very important equation for studying the zeros of $\zeta(s)$ on the critical line at 1/2 for two major reasons[9]:

Proposition 1.

1. $Z(t) = 0 \iff \zeta(1/2 + it) = 0$
2. $Z : \mathbb{R} \to \mathbb{R}$

These give a convenient way to find zeros on the critical line because real valued functions are much easier to evaluate, especially in terms of large scale computations. This greatly simplifies the area in which an algorithm must search. In a way, it looks along the real line and finds places where the Hardy function changes sign to determine if a zero can exist there. We now look to prove these two, with some of the proofs adapted from van der Meer[9].

We begin with 1. Note first that

$$\left|\frac{\Gamma(1/4+it/2)}{|\Gamma(1/4+it/2)|}\right| = \frac{|\Gamma(1/4+it/2)|}{|\Gamma(1/4+it/2)|} = 1$$

In addition

$$|\pi^{-it/2}| = \pi^{Re(-it/2)} = \pi^0 = 1$$

Now, we look at $|Z(t)|$

$$|Z(t)| = \left|\pi^{-it/2}\frac{\Gamma(1/4+it/2)}{|\Gamma(1/4+it/2)|}\zeta(1/2+it)\right| = |\pi^{-it/2}|\left|\frac{\Gamma(1/4+it/2)}{|\Gamma(1/4+it/2)|}\right||\zeta(1/2+it)|$$

but since $|\pi^{-it/2}| = \left|\frac{\Gamma(1/4+it/2)}{|\Gamma(1/4+it/2)|}\right| = 1$ we see

$$|Z(t)| = |\zeta(1/2+it)| \tag{11}$$

Thus we have $\zeta(1/2+it) = 0 \implies Z(t) = 0$. We now look to show the reverse direction. It is clear that $\pi^{-it/2} = e^{\frac{-it}{2}log(\pi)}$ is non zero for all t, as it lies on the unit circle. We just aim to show $\Gamma(1/4+it/2)$ is non zero (as this ensures no asymptotes or other zeros of Z(t)). The Gamma function is the extension of the factorial to the complex plane. For $Re(s) > 0$ it is defined as $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$. To prove that it has no zeros, it is sufficient to prove that the reciprocal of this function is entire (holomorphic on the complex plane) which we will show below. Note that this proof is done by the author (although he had to look up how to use u-substitution again). Let's start by proving the validity of the Weierstuass definition of the Gamma function:

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{k=1}^{\infty}\left[(1+\frac{s}{k})^{-1}e^{\frac{s}{k}}\right] = \lim_{n\to\infty}\frac{e^{-\gamma s}}{s}\prod_{k=1}^{n}\left[(1+\frac{s}{k})^{-1}e^{\frac{s}{k}}\right] \tag{12}$$

Where γ is the Euler-Mascheroni constant defined as $(\sum_{k=1}^{\infty}\frac{1}{k}) - log(n)$. So we see continuing from the line above:

$$= \lim_{n\to\infty}\frac{e^{-s((\sum_{k=1}^{n}\frac{1}{k})-log(n))}}{s}\prod_{k=1}^{n}\left[(1+\frac{s}{k})^{-1}e^{\frac{s}{k}}\right] = \lim_{n\to\infty}\frac{e^{slog(n)}}{se^{(s\sum_{k=1}^{n}\frac{1}{k})}\prod_{k=1}^{n}(1+\frac{s}{k})e^{\frac{-s}{k}}}$$

Now taking the product of the e terms out of the product notation:

$$\lim_{n\to\infty}\frac{e^{slog(n)}}{se^{(s\sum_{k=1}^{n}\frac{1}{k})}e^{(-s\sum_{k=1}^{n}\frac{1}{k})}\prod_{k=1}^{n}(1+\frac{s}{k})} = \lim_{n\to\infty}\frac{n^s}{s\prod_{k=1}^{n}(1+\frac{s}{k})}$$

multiplying by n!/n!:

$$\lim_{n\to\infty}\frac{n^s}{s\prod_{k=1}^{n}(1+\frac{s}{k})}\frac{n!}{n!} = \lim_{n\to\infty}\frac{n^s n!}{s\prod_{k=1}^{n}(k+s)} = \lim_{n\to\infty}\frac{n^s n!}{\prod_{k=0}^{n}(k+s)} \tag{13}$$

So we continue by showing this agrees with $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$. First consider:

$$\lim_{n\to\infty}\int_0^n\left(1-\frac{x}{n}\right)^n x^{s-1}dx \tag{14}$$

Since the integral in our first definition of gamma converges for $Re(s) > 1$, we see the integrand is measurable. In addition, $\left(1-\frac{x}{n}\right)x^{s-1}$ is dominated by $e^{-x}x^{s-1}$. Thus, by Lebesgue dominated convergence theorem, we have:

$$\lim_{n\to\infty}\int_0^n\left(1-\frac{x}{n}\right)^n x^{s-1}dx = \int_0^\infty\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n x^{s-1}dx = \int_0^\infty e^{-x}x^{s-1}dx$$

So we see (14) agrees with our definition of Gamma. Let's show that (14) also agrees with (13) and thus (12). We proceed by induction, showing for all n:

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx = \frac{n^s n!}{\prod_{k=0}^n (k+s)} \tag{15}$$

For the base case, $n = 1$:

$$\int_0^1 (1-x)\, x^{s-1} dx = \left(\frac{x^s}{s} - \frac{x^{s+1}}{s+1}\right)\Bigg|_{x=0}^1 = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$$

and we see the base case is proven. Now for the inductive step we assume (15) is true for $n-1$ and consider:

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx$$

Substituting $u = x/n$:

$$= \int_0^1 (1-u)^n (nu)^{s-1} n du = n^s \int_0^1 (1-u)^n u^{s-1} du$$

Using integration by parts we see:

$$= n^s \left(\frac{(1-u)^n u^s}{s} + n \int \frac{(1-u)^{n-1} u^s}{s} du\right)\Bigg|_{u=0}^1 = \frac{n^{s+1}}{s} \int_0^1 \frac{(1-u)^{n-1} u^s}{s} du$$

Now subbing in $v = (n-1)u$:

$$= \frac{n^{s+1}}{s} \int_0^{n-1} \left(1 - \frac{v}{n-1}\right)^{n-1} \left(\frac{v}{n-1}\right)^{(s+1)-1} \frac{dv}{n-1}$$

$$= \frac{1}{s} \frac{n^{s+1}}{(n-1)^{s+1}} \int_0^{n-1} \left(1 - \frac{v}{n-1}\right)^{n-1} v^{(s+1)-1} dv$$

Now, using our inductive hypothesis we have:

$$= \frac{1}{s} \frac{n^{s+1}}{(n-1)^{s+1}} \frac{(n-1)^{s+1}(n-1)!}{\prod_{k=0}^{n-1}(k+s+1)} = \frac{n^{s+1}(n-1)!}{s\prod_{k=1}^{n}(k+s)} = \frac{n^s n!}{\prod_{k=0}^{n}(k+s)}$$

and so we see the inductive step holds. Thus (15) holds for all $n \in \mathbb{N}$ and:

$$\Longrightarrow \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx = \lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx = \lim_{n\to\infty} \frac{n^s n!}{\prod_{k=0}^n (k+s)}$$

$$= \frac{e^{-\gamma s}}{s} \prod_{k=1}^{\infty} \left[(1 + \frac{s}{k})^{-1} e^{\frac{s}{k}}\right]$$

So we see that the Weierstrauss definition for Γ is valid. If we now consider the reciprocal of this:

$$s e^{\gamma s} \prod_{k=1}^{\infty} \left[(1 + \frac{s}{k}) e^{\frac{-s}{k}}\right] \tag{16}$$

We see that for $s = 0$ it is 0. Also if $s = -n, n \in \mathbb{N}$, then when $k = n$ $\frac{s}{k} = -1$. Thus $1 + \frac{-n}{n} = 0$ and we see that (16) evaluates to 0 here too. If s is not a non-positive integer, it is clear that (16) is non-zero. So we conclude that the non-positive integers are the only zeros of this function and are able to apply the Weierstrauss factorization theorem [6]. This gives that the function is entire as needed. Since (16) is entire, it has no poles. Thus $\Gamma(s)$ has no zeros. $\square$

With this result, we see the reverse direction of 1 from the proposition is proven. We now look to prove 2 by showing $Z(t) = \overline{Z(t)}$. This is adapted from van der Meer's paper [9]. First we will rewrite Z using the functional equation (6) defined by Riemann:

$$\zeta(s) = \frac{\xi(s)}{\Pi(\frac{s}{2})(s-1)\pi^{-s/2}} = \frac{\xi(s)}{\frac{1}{2}s(s-1)\Gamma(s/2)\pi^{-s/2}}$$

We let $s = 1/2 + it$ and substitute this expression in for $\zeta(1/2+it)$ in the definition of Z(t):

$$Z(t) = \pi^{-it/2}\frac{\Gamma(1/4+it/2)}{|\Gamma(1/4+it/2)|}\frac{\xi(1/2+it)}{\frac{1}{2}(1/2+it)(-1/2+it)\Gamma(1/4+it/2)\pi^{-(1/4+it/2)}}$$

$$= \frac{1}{|\Gamma(1/4+it/2)|}\frac{\xi(1/2+it)}{\frac{1}{2}(1/2+it)(-1/2+it)\pi^{-1/4}} = \frac{-2}{|\Gamma(1/4+it/2)|}\frac{\xi(1/2+it)}{(1/4+t^2)\pi^{-1/4}} \tag{17}$$

The only possible source of non-real values for $Z(t)$ is $\xi(1/2+it)$. Thus:

$$Z(t) = \overline{Z(t)} \iff \xi(1/2+it) = \overline{\xi(1/2+it)}$$

Let's now show $\xi(1/2+it) = \overline{\xi(1/2+it)}$. This follows if $\overline{\xi(1/2+it)} = \xi(\overline{1/2+it})$ since $\xi(s) = \xi(1-s)$.

$$\overline{\xi(1/2+it)} = \overline{\Pi(1/4+it/2)} \cdot \overline{(-1/2+it)} \cdot \overline{\pi^{-(1/4+it/2)}} \cdot \overline{\zeta(1/2+it)}$$

By lemma 2.15 from the van der Meer paper[9]:

$$= \Pi(\overline{1/4+it/2})(-1/2-it)\pi^{\overline{-(1/4+it/2)}}\zeta(\overline{1/2+it}) = \xi(\overline{1/2+it})$$

By Riemann's result that $\xi(s) = \xi(1-s)$:

$$= \xi(1/2-it) = \xi(1-(1/2-it)) = \xi(1/2+it)$$

Thus we see $\overline{\xi(1/2+it)} = \xi(\overline{1/2+it})$ and it follows that $Z(t) = \overline{Z(t)}$. So the proposition is proven. Also, from (17) we see that $Z(t)$ is continuous since:

- The reciprocal of Γ is entire
- $(1/4+t^2)\pi^{-1/4} \neq 0 \forall t \in \mathbb{R}$
- ξ is analytic in the critical strip

With a better understanding of the Hardy function and the nice properties it has, we can see why it is so useful in finding zeros of ζ. By looking for the change of signs on this continuous real function we can find the zeros of the Riemann zeta function on the critical line at 1/2. At first glance, this does not seem to be useful in verifying the Riemann hypothesis. How can one be sure that in looking on this line one is finding all the zeros? An important question that brings us to our next section.

2.3 Algorithms

The way one can verify the RH for such large numbers of zeros is by developing algorithms that can test for the existence of zeros. It is difficult for algorithms to search an area of the complex plane so it is common to focus exclusively on the critical line at 1/2 using methods such as evaluation of the Hardy function. However, Turing devised a method for ensuring that all zeros in an area are found, given that "enough" zeros in that area are found to begin with [3].

It is known that for a critical line σ between 0 and 1 and $0 < t < T$, the number of zeros of $\zeta(\sigma + it)$ is:

$$N(t) = 1 + \frac{\theta(T)}{\pi} + S(T)$$

Where $\theta(x) = \sum_{p \leq x} log(p)$, $S(T) = \frac{1}{\pi} arg\zeta(\frac{1}{2} + it)$. Now, a Gram point g_n is a solution > 7 to

$$\theta(g_n) = n\pi$$

and a regular Gram point is one such that $S(g_n) = 0$. "One of the key point in RH verification is the ability to find regular Gram points. Once regular Gram points are found, it suffices to check that between them, the expected number of change of sign of Z(t) occurs, in order to numerically check the RH in this zone" [3]. Turing's method is slightly beyond this text. Nonetheless, I will give a high level description paraphrased from Gourdon:

When searching for zeros of the Hardy function, one can find a sequence (h_n) such that:

- $(-1)^n Z(h_n + g_n) > 0$ (where g_n is a gram point described above)
- $(h_n + g_n)$ is increasing
- h_n sufficiently small, zero when possible

If h_n sufficiently close to $h_m = 0$ for n near m, then g_m is a regular gram point. Turing was able to find a bound on $S(g_m)$ that when certain conditions were met, $-2 < S(g_m) < 2$. The paper by Gourdon claims that $S(g_m)$ must be an even integer and thus when $-2 < S(g_m) < 2$, $S(g_m) = 0$, giving g_m to be a regular gram point. By looking for regular gram points while searching the Hardy function one can greatly improve the efficiency of verification algorithms (as opposed to other methods that do not use the Hardy function to find regular Gram points).

It is in this way that Gourdon was able to verify the hypothesis for the first 10^{13} zeros. He used the idea from the Odlyzko-Schönhage algorithm to evaluate the Hardy function efficiently in a range of $T \leq t \leq T + \Delta$ with $\Delta = O(\sqrt{T})$. They define $Z(t)$ in this interval as:

$$Z(t) = \sum_{n=1}^{k_0 - 1} \frac{cos(\theta(t) - tlog(n))}{\sqrt{n}} + Re(e^{-i\theta(t)} F(t)) + \sum_{n=k_1+1}^{m} \frac{cos(\theta(t) - tlog(n))}{\sqrt{n}} + R(t)$$

Where $R(t)$ is the remainder term and $F(t)$ is defined as:

$$F(t) = F(k_0 - 1, k_1, t) := \sum_{k=k_0}^{k1} \frac{1}{\sqrt{k}} e^{itlog(k)}$$

Much of the work goes into methods for evaluating $F(t)$ efficiently so that zeros of $Z(t)$ and regular Gram points can be found quickly. The above overview is brief and provides a mere glimpse into the work done by Gourdon and Patrick Demichel "who managed the distribution of the computation on several computers in order to make the RH verification on the first 10^{13} zeros possible" [3]. However, the reader might find the simple description sufficient to illustrate how the Hardy function is put to use in major computations.

2.4 Some Computations

Since the methods and results described above require a great deal of time and computing power to implement, we set our sights a little lower when demonstrating computations. We can plot the Hardy function in a relatively small domain and use software to search for a sign change. This method will compute the zeros of the zeta function in this portion of the critical line, but does not verify the Riemann hypothesis (as we are not implementing Turing's or any similar method for ensuring all zeros are calculated). To further reduce the area in which we search, we can use the result from lemma 2.15 of van der Meer [9]:

$$\overline{\zeta(1/2+it)} = \zeta(\overline{1/2+it})$$

So if $zeta(1/2+ix) = 0, x \in \mathbb{R}$ then:

$$0 = \overline{\zeta(1/2+ix)} = \zeta(\overline{1/2+ix})$$

and we get that zeros on the critical line come in conjugate pairs. So when we plot the Hardy function, we can look for non-negative t and we will get zeros for negative t as well. Below is a some code and the plot of the Hardy function it outputs, as well as a table containing the location of zeros found on the plot. The code is:

```
import numpy as np
import matplotlib.pyplot as plt
# The next library contains the
# zeta(), zetazero(), and siegelz() functions
from mpmath import *

mp.dps = 25;
mp.pretty = True


def graph_zeta(real, image_name):
    A, B, C = [], [], []

    fig = plt.figure()
    ax = fig.add_subplot(111)

    for i in np.arange(0.1, 100.0, 0.1):
        function = zeta(real + 1j * i)
        function1 = siegelz(i)
        A.append(abs(function))
        B.append(function1)
        C.append(i)

    ax.grid(True)
    ax.plot(C, A, label='modulus of Riemann zeta function along \
    critical line, s = 1/2 + it', lw=0.8)
    ax.plot(C, B, label='Riemann-Siegel Z-function, Z(t)', lw=0.8)
    ax.set_title("Riemann Zeta function - Re(s)=1/2")
    ax.set_ylabel("Z(t)")
    ax.set_xlabel("t")
```

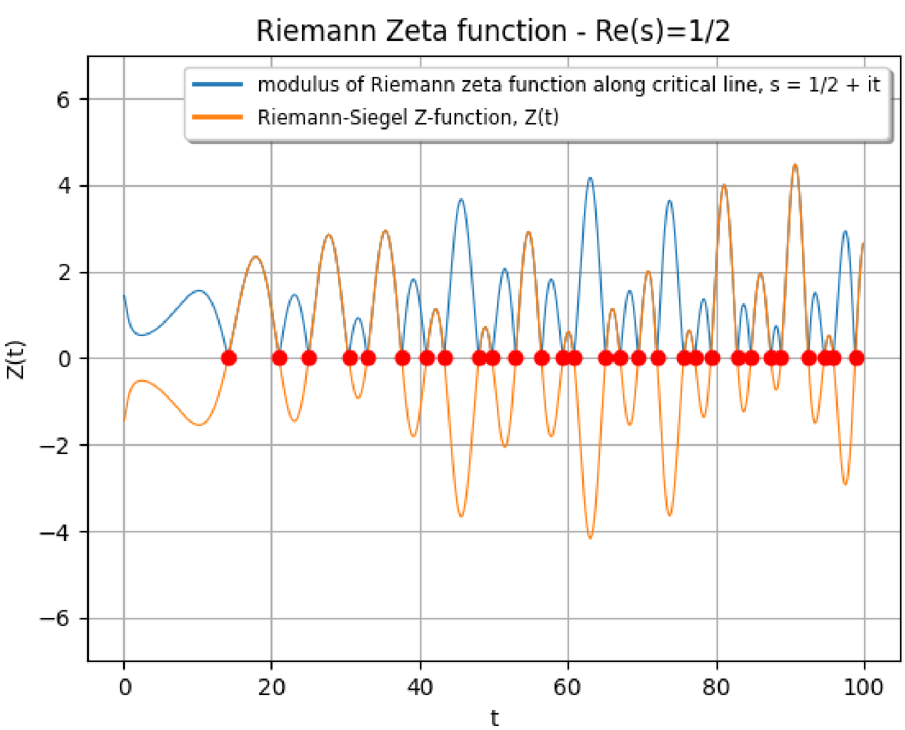
```
    # Include legend
    leg = ax.legend(shadow=True)
    # Edit font size of legend to make it fit into chart
    for t in leg.get_texts():
        t.set_fontsize('small')
    # Edit the line width in the legend
    for l in leg.get_lines():
        l.set_linewidth(2.0)
    # Plot the zeroes of zeta

    for i in range(1, 30):
        zero = zetazero(i)
        ax.plot(zero.imag, [0.0], "ro")

    # save plot and print that it was saved
    ax.set_ylim(-7, 7)
    plt.savefig(image_name)
    print("Successfully plotted %s !" % image_name)
    plt.close()


graph_zeta(0.5, "Z(t)_Plot.png")
```

This code is provided by Daner Ferhadi's GitHub page which has some open source methods for evaluating the Hardy function[2]. It outputs a graph of $Z(t)$ with the red dots at zeros (which by our work are also zeros of $\zeta(1/2 + it)$). Note that one can extend the range we are looking in by modifying the middle argument in np.arrange in line 17 and changing the range of the for loop under the "# Plot the zeros of zeta" comment. As is, it gives the plot below:



The following table is from the van der Meer paper which shows the zeros in the same range as the plot above[9].

range of t	zero		range of t	zero		range of t	zero
(10,20)	14.13472514		(51,55)	52.97032148		(78,80)	79.33737502
(20,23)	21.02203964		(55,58)	56.44624770		(80,83)	82.91038085
(23,30)	25.01085758		(58,60)	59.34704400		(83,85)	84.73549298
(30,32)	30.42487613		(60,64)	60.83177852		(85,88)	87.4252746
(32,35)	32.93506159		(64,66)	65.11254405		(88,90)	88.80911121
(35,40)	37.58617816		(66,68)	67.07981053		(90,93)	92.49189927
(40,42)	40.91871901		(68,70)	69.54640171		(93,95)	94.65134404
(42,45)	43.32707328		(70,75)	72.06715767		(95,97)	95.87063423
(45,49)	48.00515088		(75,76)	75.70469070		(97,100)	98.83119422
(49,51)	49.77383248		(76,78)	77.14484007			

"Zeroes of Z(t) in the interval (0,100) calculated using Maple" [9].

This is a small glimpse into the world of calculating zeros of the zeta function. Our evaluation of $Z(t)$ is quite simple and inefficient compared to the modern methods used to verify the RH for large magnitudes. Indeed, it does not even verify that these are the only zeros in their respective areas of the critical strip (using methods such as Turing's from the last section). One may yet find it to be a satisfying exercise considering the work done above to understand these simple computations.

3 Concluding Remarks

This project has explored the Riemann hypothesis and the numerical verification of it. Using $Z(t)$, we can simplify the problem to a continuous real function. The value of these computations is important for understanding the distribution of prime numbers and this is why it is of concern to number theorists. This numerical verification is fascinating. It provides a unique challenge of creating efficient ways to evaluate an intricate function as a form of evidence for mathematical conjecture.

The intersection of computer science and pure mathematics here allows for constant improvements and interest for many fields of study. The RH is a Millennium Prize Problem and will continue to attract the attention of many aspiring mathematicians with it's complexity, beauty, and consequences. One may find the overview in this paper to spark interest; the author certainly has.

References

[1] E. Bombieri. Problems of the millennium: the riemann hypothesis. http://claymath.org/sites/default/files/official_problem_description.pdf.

[2] Daner Ferhadi. Riemann-zeta. Code for displaying and computing zeros of Hardy Function.

[3] Xavier Gourdon. The 10^{13} first zeros of the riemann zeta function, and zeros computation at very large height. October 24th 2004. http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf.

[4] David Jekel. The riemann zeta function. 2013. https://sites.math.washington.edu/~morrow/336_13/papers/david.pdf.

[5] George Jin. Prime number theorem. *Research Gate*, 2021. https://www.researchgate.net/publication/349054409_Prime_Number_Theorem_history_and_two_different_proofs.

[6] Encyclopedia of Mathematics. Weierstrass theorem. https://encyclopediaofmath.org/wiki/Weierstrass_theorem.

[7] David Platt and Timothy Trudgian. The error term in the prime number theorem. *University of Bristol, University of New South Wales Canberra*, 2021. https://arxiv.org/pdf/1809.03134.pdf.

[8] Bernhard Riemann and David R. Wilkins. On the number of prime numbers less than a given quantity. *(Translated in 1998 by Wilkins)*, 1859. https://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf.

[9] R. van der Meer. Zeros of the zeta function: Bachelor's project mathematics. *University of Groningen*, 2020. https://fse.studenttheses.ub.rug.nl/21726/1/bMATH_2020_MeerRvander.pdf.