Transcendence of e and π

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1 Introduction

1.1 e and π

e and π are one of the most cryptic numbers in Mathematics. They somehow appear in all mathematics fields, ranging from infinitesimal calculus to algebraic geometry. Mathematicians have been obsessed with π since the dawn of time, they tried to find uses for π, to calculate π up to decimal places, and so on... Compared to π which has been known for almost 4000 years by Ancient Babylonians, e is a new-born in mathematics, first discoveries of the number date back $\sim$ 500 years ago, which intrigued mathematicians as much as his peer π. e is also known as Napier's constant, or Euler's number, named after Leonhard Euler, to whom we owe many important mathematical results, in which he has extensively used e and π, and popularized their symbol representation as the Latin letter e and the greek letter π.[1]

These seemingly unrelated numbers come in fact together in many mathematical equations, most notably in Euler's identity: $e^{i\pi} + 1 = 0$, which is considered the most beautiful equation in Mathematics[2], $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, and in many other numerous occasions...

e and π are irrational numbers, as in they can't be represented as a fraction. More than that, e and π are <u>transcendental</u> numbers. In this paper, I will undertake the irrational (no pun intended) enterprise of proving that e and π are transcendental.

[1] A Brief History of Pi (π) | Exploratorium.
[2] Terence Tao

1.2 History of transcendence proofs

The subject of transcendence is a recent one, mathematicians suspected that e and π are different from other real numbers, and the term "transcendence" was first used by Leibniz in 1682, where he proved that the function $\sin x$ was not algebraic, as in, some outputs, like $\sin(1)$ are transcendental numbers. Euler was the first to give a modern definition of a transcendental number in the 18th century.

Joseph Liouville proved the existence of transcendental numbers in 1844 and defined the "Liouville numbers", which are artificial numbers he defined to be "almost" rational, but transcendental. Liouville conjectured that e and π were transcendental, but couldn't prove it.

Charles Hermite, in an attempt to prove that π is transcendental, proved first that e is transcendental in 1873, but was missing one last crucial step. He wrote in a letter to his friend:

> I shall risk nothing on an attempt to prove the transcendence of the number π. If others undertake this enterprise, no one will be happier than I at their success, but believe me, my dear friend, it will not fail to cost them some effort.[3]

And he was right, 9 years later, the mathematician Ferdinand von Lindemann, gave the first proof of transcendence of π, greatly inspired by Hermite's work. In 1893, David Hilbert (no, he was not writing this in a hotel) came up with a more refined version of these proofs of transcendence.[4]

2 Problem

2.1 Transcendence

What is a transcendental number? Here is a basic definition:

Definition 2.1. *A transcendental number is a number that is not algebraic.*

This shy of a definition motivates us to define algebraic numbers, which are as follows:

[3] Charles Hermite; letter to C.W. Borchardt

[4] George F. Simmons, Calculus Gems, Chapter A29-A30

Definition 2.2. *An algebraic number is a number that is a root of a polynomial with integer (or rational) coefficients.*

Interestingly enough, transcendental numbers don't just timidly exist, Cantor proved that there is a vast amount of transcendental numbers. Take the real line, and assume you randomly pick a number from this line. Well, this randomly picked is "almost surely" (i.e probability of the event is 1) a transcendental number. Most real numbers are thus transcendental.[5]

Liouville proved the existence of these numbers, using the Liouville numbers he defined as such:

Definition 2.3. *a Liouville number is a real number x such that, $\forall n \in \mathbb{Z}$, $\exists(p, q)$ with $q > 1$ such that:*

$$0 < |x - \frac{p}{q}| < \frac{1}{q^n}$$

One thing Liouville realized, is that his newly defined numbers can be well approximated using rational numbers, which we can see by the way they are defined. Another thing is that this is not restricted to Liouville numbers, but transcendental numbers in general. Even more, it is not easy to approximate irrational algebraic numbers by rational numbers. This seems counter-intuitive, because we know so little of transcendental numbers, and we know way more about irrational numbers, yet we can approximate too well transcendental numbers and not irrational algebraic numbers... What does this have to do with the subject of our paper? Well, this prompts us to think that one way of proving that e and π are transcendental is by proving they're not algebraic irrational numbers, using approximation.[6]

So, all of this means that in order to prove that e and π are transcendental numbers, we would need to prove that they are not algebraic, or in other words, that there exists no single-variable polynomial f with integer or rational coefficients, such that e or π is a root of said polynoomial.

But a problem arises here, we saw that transcendental numbers can be approximated too well using rational numbers, and rational numbers are roots of polynomials, so if e is very close to a rational number, and this rational number is a root of some polynomial f, then $f(e) \approx 0$ i.e very close to 0. But, we can also find another polynomial g, such that $g(e)$ is even closer

[5] Vo, Thanh, and Huan. n.d., INTRODUCTION to TRANSCENDENTAL NUMBERS.
[6] Wikipedia, Liouville number

to 0 than $f(e)$ and so on... So where do draw the line between $f(e) = 0$ and $f(e) \approx 0$? Even with all the computational power we have today, we still will always have computational limitations, as depending on the relative error we work with, we can always find a polynomial with rational coefficients such that $f(e)$ seems (in paper) equal to zero, while in fact, it's just very, very, very close to it.

2.2 Proofs

I've spent a lot of time trying to come up with a good idea to prove these results. I was trying to prove that π is transcendental but in vain. My intuition was that π is a more common number in mathematics in general, and that we have so many theorems and results that are related to π, but I soon realized that I couldn't even come up with proof that π is irrational!

On the other hand, I could see that e is irrational by just using its sum representation ($\sum_{n=0}^{\infty} \frac{1}{n!}$)and finding that it can't solve any linear polynomial using contradiction. Of course, π also has a neat sum representation, as:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

but this sum wasn't really helpful as I couldn't exploit some properties of the factorial that I previously used in the representation of e as a sum.

Then I realized something: Nearly all proofs of the transendence of π used e's transcendence, which was comparable to the difficulty irrationality of e and that of π. In that regard, let's try to prove that e is transcendental first.

Theorem 2.1. *e is transcendental.*

The following proof is derived from two proofs[7][8] that I found to be the farthest from other proofs, which were all just variations of Hermite's original 1873 proof.

Proof. We start with the oldest trick in the book of mathematics, the proof by contradiction, we suppose that e is not transcendental, so there exists a polynomial:

[7] Ross, Marty, e and π are transcendental. 2018.

[8] Michael Spivak. Calculus. Cambridge university press, third edition,2006

$$P(x) = a_0 + a_1 x + ... + a_n x^n$$

such that $P(e) = 0$, or:

$$a_0 + a_1 e + ... + a_n e^n = 0$$

for some $n \in \mathbb{Z}$, and $\{a_0, a_1, a_2, ..., a_n\}$ integer coefficients. We will show that, in particular, powers of e can be written in the form:

$$e^k = \frac{N_k + \epsilon_k}{N}, \text{ for } k = 1, 2, ..., n$$

Let p an arbitrary (for now) large prime, define:

$$f_p(x) = x^{p-1}[(x-1)(x-2)...(x-n)]^p$$

We define N, N_k and ϵ_k:

$$\begin{cases} N = \frac{1}{(p-1)!} \int_0^\infty e^{-x} f_p(x) dx \\ N_k = \frac{1}{(p-1)!} \int_k^\infty e^{k-x} f_p(x) dx \\ \epsilon_k = \frac{1}{(p-1)!} \int_0^k e^{k-x} f_p(x) dx \end{cases}$$

The choice of N, N_k and ϵ_k aligns with our equation, as:

$$\begin{aligned} e^k &= \frac{e^k N}{N} \\ &= \frac{N_k + \epsilon_k}{N} \end{aligned}$$

Before proceeding, let's have a look at the gamma function, which will be useful for our proof:

Definition 2.4. *The gamma function Γ is defined as such for any positive integer n:*

$$\Gamma(n) = (n-1)!$$

and more generally, for any complex number z with $\Re(z) > 0$:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

So one basic property we can extract from this is that:

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t} dt = n!$$

If we expand our function $f_p(x)$, we get:

$$\begin{aligned} f_p(x) &= x^{p-1}[(x-1)(x-2)...(x-n)]^p \\ &= x^{p-1}(x^n + \cdots \pm n!)^p \\ &= x^{p-1}(x^{np} + \cdots \pm (n!)^p) \\ &= x^{p-1+np} + \cdots \pm x^{p-1}(n!)^p \end{aligned}$$

So:

$$f_p(x) = \sum_{j=0}^{np} n_j x^{p-1+j}$$

where $m_0 = \pm(n!)^p$. Replacing in N:

$$\begin{aligned} N &= \frac{1}{(p-1)!} \int_0^\infty e^{-x} \sum_{j=0}^{np} n_j x^{p-1+j} dx \\ &= \sum_{j=0}^{np} \frac{n_j}{(p-1)!} \int_0^\infty x^{p-1+j} e^{-x} dx \\ &= \sum_{j=0}^{np} \frac{n_j}{(p-1)!} \Gamma(p+j) = \sum_{j=0}^{np} \frac{n_j (p-1+j)!}{(p-1)!} \\ &= m_0 + \sum_{j=1}^{np} \frac{n_j (p-1+j)!}{(p-1)!} \end{aligned}$$

All other terms besides for $j = 0$ are of the forms $\frac{n_j(p-1+j)!}{(p-1)!}$, starting from $\frac{n_j(p-1+1)!}{(p-1)!} = \frac{n_j(p)!}{(p-1)!} = n_j p$, and so on, you can see that all of those terms are multiples of p, and $m_0 = \pm(n!)^p$ is not divisible by p, as p is a large prime, thus $p > n$, so $n!$ can't be divisible by p, same for m_0.

So:

$$N \equiv m_0 \mod p$$

So, N is not divisible by p. $\boxed{1}$

Now, similarly for N_k:

$$N_k = \frac{1}{(p-1)!} \int_k^{\infty} e^{k-x} f_p(x) dx$$

let $t = x - k$, we have $dt = dx$, and note that:

$$f_p(t+k) = (t+k)^{p-1}[(t+k-1)(t+k-2)...(t+k-n)]^p$$

$k = 1, 2, \ldots, n$, so the degree of $f_p(t+k)$ is between $np + p - 1$ and p, so we can write it as:

$$f_p(t+k) = \sum_{j=1}^{np} n_j t^{p-1+j}$$

We substitute:

$$\begin{aligned} N_k &= \frac{1}{(p-1)!} \int_k^{\infty} e^{-t} f_p(t+k) dt \\ &= \frac{1}{(p-1)!} \int_k^{\infty} e^{-t} \sum_{j=1}^{np} n_j t^{p-1+j} \\ &= \sum_{j=1}^{np} \frac{n_j (p-1+j)!}{(p-1)!} \end{aligned}$$

So N_k is a sum of multiples of p, so N_k is divisible by p. $\boxed{2}$

Now:

$$
\begin{aligned}
\epsilon_k &= \frac{1}{(p-1)!}\int_0^k e^{k-x} f_p(x)dx \\
&\leq \frac{1}{(p-1)!}\int_0^k e^{k-x} \max_{x\in[0,n]} f_p(x)dx \\
&\leq \frac{1}{(p-1)!}\int_0^k Ae^{k-x}dx \leq \frac{Ae^n}{(p-1)}(1-e^{-k}) \\
&\leq \frac{Ae^n}{(p-1)}
\end{aligned}
$$

So ϵ_k can be made as small as we want relative to the p we choose. $\boxed{3}$
Now, let's get back to our main proof, we have $e^k = \frac{N_k+\epsilon_k}{N}$, so:

$$
\begin{aligned}
&a_0 + a_1 e + ... + a_n e^n = 0 \\
&a_0 + a_1(\frac{N_1+\epsilon_1}{N}) + ... + a_n(\frac{N_n+\epsilon_n}{N}) = 0 \\
&a_0 N + a_1(N_1+\epsilon_1) + ... + a_n(N_n+\epsilon_n) = 0 \\
&(a_1N_1 + \cdots + a_nN_n) + (a_0N + a_1\epsilon_1 + \cdots + a_n\epsilon_n) = 0 \\
&a_1N_1 + \cdots + a_nN_n = -(a_0N + a_1\epsilon_1 + \cdots + a_n\epsilon_n)
\end{aligned}
$$

By $\boxed{3}$, $(a_1\epsilon_1+\cdots+a_n\epsilon_n) \to 0$ as $p \to \infty$, and by $\boxed{2}$, a_0N is not a multiple of p. So the expression $a_0N + a_1\epsilon_1 + \cdots + a_n\epsilon_n$ will not be a multiple of p given p large enough.

On the other hand, by $\boxed{2}$, $a_1N_1 + \cdots + a_nN_n$ will be a multiple of p, so we've come to a contradiction $\Rightarrow\Leftarrow$. So our original assumption that e is not transcendental is false.

Thus, we proved that e is a transcendental number. □

Theorem 2.2. *π is transcendental.*

Proof. Let's move to the proof of transcendence of π now. We could use the Lindemann-Weierstrass theorem, which states that:[9]

Theorem 2.3 (Lindemann-Weierstrass theorem)**.** *If $a_1, \ldots, a_n$ are algebraic numbers that are linearly independent over the rationals, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent over $\mathbb{Q}$.*

[9]Wikipedia, Lindemann–Weierstrass Theorem.

Corollary 2.3.1. *e^a is transcendental for a algebraic.*

Proof. Let a be a non-zero algebraic number, then $\{a\}$ is a linearly independent set over the rationals, so by our theorem, $\{e^a\}$ is an algebraically independent set. In particular, a one-element set is algebraically independent if and only if that element is transcendental. So e^a is transcendental for a algebraic. $\square$

For the sake of contradiction, suppose π is algebraic, then there exists a polynomial $p(x)$ such that: $p(\pi) = 0$. So $i\pi$ is a root of the polynomial $p(ix)p(-ix)$. Hence, $i\pi$ is also algebraic. From our corollary, this means that $e^{i\pi}$ is transcendental. Remember our most beautiful mathematical equation? $e^{i\pi} + 1 = 0$, so $e^{i\pi} = -1$, which is definitely not transcendental $\Rightarrow\Leftarrow$. So, we proved by contradiction that π is transcendental.

$\square$

The Lindemann-Weierstrass theorem is one of the most powerful results in transcendence theory. For the first time, we were able to prove whether some numbers are transcendental or not, like π in this example, e (taking the special case where the power of e is 1), the logarithm $\ln(a)$ (for a algebraic) function, and one of my favorite functions: the Lambert $W(a)$ function, and so on...

I've estimated that it would be too bothersome for me to try to prove this theorem, as I would have to prove a generalized theorem that is used in many results just to "downgrade" and take special cases to solve our transcendence proofs...

An alternative way to prove transcendence of π:

Proof. Suppose π is transcendental. then there exists a polynomial $p(x)$ such that: $p(\pi) = 0$. So $i\pi$ is a root of the polynomial $p(ix)p(-ix) = q(x)$. Let $q(x) = a_0 + a_1x + ... + a_nx^n$. Suppose $\alpha_1, ...\alpha_n$ roots of this polynomial. Then $1 + e^{\alpha_i} = 0$ for $i \in \{1, .., n\}$. So:

$$(1 + e^{\alpha_1}) + (1 + e^{\alpha_2}) + \cdots + (1 + e^{\alpha_n}) = 0$$

Expanding this gives us:

$$k + e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_r} = 0$$

for some positive k. Now, in a similar fashion to our proof for e (more advanced though) that we won't cover here, we can prove that such an equation is impossible. Remember, we showed that the equation $a_0+a_1e+...+a_ne^n=0$ has no solution, ultimately proving transcendence of π. $\square$

2.3 Product and linear combinations

Now that we found some transcendental numbers, can we find more? Yes, we can find as many additional transcendental numbers as we want. Applying any non-constant single-variable algebraic function to a transcendental argument yields a transcendental number.

Theorem 2.4. *If $a = f(\pi)$, with f being an algebraic function, then a is a transcendental number.*

Proof. This is thanks to the properties of algebraic numbers. Assume $f(\pi)$ is not transcendental, i.e algebraic. And we know that the algebraic property of numbers is conserved under sum, multiplication, and inverse. So by means of applying other algebraic functions to $f(\pi)$ (examples: applying $f(x) = \frac{1}{\sqrt{x}}$ to $\frac{1}{\pi^2}$, $f(x) = \frac{x}{6}$ to $6\pi, \dots$) in the following equation:

$$a_0 + a_1 f(\pi) + a_2 f(\pi)^2 + ... + a_n f(\pi)^n = 0$$

We can end up with this equation:

$$b_0 + b_1\pi + \cdots + b_n\pi^n = 0$$

Which is impossible, as π is transcendental. So, by contradiction, $a = f(\pi)$ is transcendental. $\square$

One result I came up with that I found very interesting, can be stated as follows:

Theorem 2.5. *At least one of $e + \pi$ and $e\pi$ must be transcendental.*

Let's prove this lemmafirst:

Lemma 2.6. *Let α be a root of a polynomial $p(x)$ with algebraic coefficients. Then α is an algebraic number.*

Proof. Let $\mathbf{K}$ the field generated by the coefficients of p. These coefficients are algebraic, so $\mathbf{K}$ is finitely generated over $\mathbb{Q}$. α is finite-dimensional over $\mathbf{K}$, so it is also finite-dimensional over $\mathbb{Q}$. In other words, there is a linear combination of powers of α that vanishes, so α is algebraic.

$\square$

Proof. Now, to the theorem. We will proceed by contradiction. Assume $e + \pi$ and $e\pi$ are both algebraic. Let $q(x) = (x - e)(x - \pi)$, then $q(x) = x^2 - (e + \pi)x + e\pi$ has algebraic coefficients. So, by our lemma, the roots of the polynomial e and π are algebraic. Contradiction, as e and π are both known to be transcendental.

So, at least one of $e + \pi$ and $e\pi$ must be transcendental. $\square$

3 Conclusion

The 19th century was a very important century for number theory. A century where brilliant minds all over the world spent unconceivable time and effort, leading to scientific breakthroughs, fueled by their lust for knowledge on these numbers that have meaning beyond our understanding, that appear in the Laws of Physics that govern our universe. This introduced us to new fields of Mathematics, like Transcendence theory and Diophantine approximation. These fields have numerous applications in class number problems, divisor properties of arithmetic sequences, or linear forms in elliptic logarithms...

In the hopes that this write-up was insightful to some extent, I will leave you with this profound quote from Paul Halmos:

> What's the best part of being a mathematician? I'm not a religious man, but it's almost like being in touch with God when you're thinking about mathematics. God is keeping secrets from us, and it's fun to try to learn some of the secrets.[10]

[10] Paul Halmos, Celebrating 50 Years of Mathematics

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