The theory of partitions

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Abstract

An introduction to the theory of integer partitions with a focus on generating functions.

1 Introduction

In 1740, french mathematician Philip Naude (1684-1747) raised the following question in a letter to Leonard Euler (1707-1783) [4]. In how many ways can an integer n be represented as a sum of integers? Euler subsequently discovered many new ideas and techniques to answer this question and thus founded the theory of partitions. His main tool in studying partitions was *generating functions* which is the focus of this short paper. By examining patterns in the first few values of the partitions function, Srinivasa Ramanujan (1887-1920) discovered some beautiful congruence properties that we will motivate. Norman Macleod Ferrers (1829-1903) discovered a useful ways of visualising integer partitions known as Ferrer's graph. This approach opens the door for elegant bijective proofs of partition identities, but we won't cover these in this paper. We will rather explore the power of generating functions in studying partitions. Many famous mathematicians have made contributions to the theory of integer partitions including Euler, Legendre, Ramanujan, Hardy, Rademacher, Sylvester, Selberg and Dyson [3]. Thus, this paper only touches a tiny part of the field.
The proofs for propositions 2.1, 4.2, 4.4, 6.1 are original. The idea for the proof of section 5.3 is the same as in [3] but the calculations are original. The code at the end of section 6 is original and can be found here [1]. Finally, section 7 is original. The rest is mostly a reformulation of the relevant sections in *Integer Partitions* by Andrews and Eriksson [3]. And section 6 is a reformulation of a paper by Andrews' [2].

1.1 Initial definitions

Definition 1. *For $\lambda_1, \lambda_2 ... \lambda_k \in \mathbb{N}$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ the multiset $\sigma = [\lambda_1, \lambda_2, ..., \lambda_k]$ is a partition of n if $\lambda_1 + \lambda_2 + ... + \lambda_k = n$.*

Definition 2. *Let $A_n = \{\sigma : \sigma \text{ is a partition of } n\}$.*

Since σ is a multiset instead of a set, multiple λ's may share the same value.

Definition 3. *Let $mult(\lambda) :=$ "the number of λ's in σ" for each $\lambda \in \sigma$.*

We will also use the notation $\Sigma\sigma = \sum_{i=1}^{k} \lambda_i = n$ as explicitly writing out the λ's will be helpful. Due to the ordering of the λ's, all permutations of σ are equivalent. The study of integer partitions explores certain properties and relationships between partitions with different properties, such as σ with only odd λ's or σ's where all λ's have multiplicity 1. We are also interested in the number of partitions for a given n, so the function $p(n) = |A_n|$.

2 Generating functions

The first goal we will explore is getting some expression for $p(n)$. We start by breaking down A_n into sets based on what the largest λ is in each partition.

Definition 4. *Let*

$$B_m(n) := \{\sigma \in A_n : max(\sigma) \leq m\} \text{ then let } b_m(n) := |B_m(n)| \quad so \quad p(n) = \lim_{m\to\infty} b_m(n)$$

$$B_m(n)^* := \{\sigma \in A_n : max(\sigma) = m\} \text{ then let } b_m(n)^* := |B_m(n)^*| \quad so \quad b_m(n) = \sum_{i=0}^{m} b_m(n)^*$$

This type of sum hints at the fact that geometric series of the form $S = \sum_{k=0}^{\infty} q^k$ will often appear. If such a series converges then we have a very neat close form solution, $S = \frac{1}{1-q}$. If we instead have a partial sum then we get $\sum_{k=0}^{N} q^k = \frac{1-q^{N+1}}{1-q}$. This motivates the use of *generating functions.*

Definition 5. *For a sequence* $(a_k) := \{a_k\}_{k=1}^{\infty}$*, we define its generating function as*

$$G_{(a_k)}(q) = \sum_{k=0}^{\infty} a_k q^k \tag{1}$$

where q *is a free variable such that the series converges.*

In $\mathbb{R}$ we must have $|q| < 1$. The trick is that from a recursive definition of (a_k), after expanding we can get terms with $G_{(a_k)}(q)$ on the right hand side. We can then solve for $G_{(a_k)}(q)$ in terms of some expression of k's and the q^k's. Then, if we can rewrite this as $G_{(a_k)}(q) = \sum_{k=0}^{\infty} c_k q^k$, for some c_k's, we must have $c_k = a_k \, \forall k$ giving us the close form expression for (a_k). We will therefore look at the generating function of $p(n)$. First we use the fact that $p(n) = \lim_{m\to\infty} b_m(n)$. So, we fix m and look at

$$G_{(b_m(n))}(q) = \sum_{n=0}^{\infty} b_m(n) q^n \tag{2}$$

For $m = 1$, we need a partition σ of n such that for all λ's, $\lambda \leq 1$ so $\sigma = [1, ..., 1]$ where we have n one's. So $\forall n$, we only have one option so $b_m(n) = 1$ and we get

$$G_{(b_1(n))}(q) = \sum_{n=0}^{\infty} 1 \cdot q^n = \frac{1}{1-q} \quad . \tag{3}$$

For $m = 2$, using the notation $\Sigma\sigma = n$ we get

$$\begin{aligned} G_{(b_2(n))}(q) &= \underbrace{q^2}_{n=2} + \underbrace{q^{2+1}}_{n=3} + \underbrace{q^{2+1+1} + q^{2+2}}_{n=4} + \underbrace{q^{2+1+1+1} + q^{2+2+1}}_{n=5} + \underbrace{q^{2+1+1+1+1} + q^{2+2+1+1} + q^{2+2+2}}_{n=6} + ... \\ &= (q^1 + q^{1+1} + q^{1+1+\cdots}) + (q^2 + q^{2+1} + q^{2+1+1} + q^{2+1+1+\cdots}) + (q^{2+2} + q^{2+2+1} + q^{2+2+1\cdots}) + ... \\ &= (q^1 + q^2 + ...) + q^2(q^1 + q^2 + ...) + q^4(q^1 + q^2 + ...) + ... \\ &= \frac{1}{1-q} + \frac{q^2}{1-q} + \frac{q^4}{1-q} + ... \\ &= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \end{aligned} \tag{4}$$

We claim that this pattern continues.

Proposition 2.1.

$$G_{(b_r(n))}(q) = \prod_{j=1}^{r} \frac{1}{1-q^j} \qquad \forall r \in \mathbb{N}_1 \tag{5}$$

Proof. We proceed by induction on r. Let

$$P(r) = \text{“} G_{(b_r(n))}(q) = \prod_{j=1}^{r} \frac{1}{1-q^j} \text{”} \tag{6}$$

we prove that $P(r) = True \ \forall r \in \mathbb{N}_1$.
Base case. $r = 1$. We just showed in equation (4) that $P(1) = True$
Induction Hypothesis. Suppose that for some $k \in \mathbb{N}_1$, $P(k) = True$.
Induction Step. We can also write the generating function as

$$\begin{aligned} G_{(b_k(n))}(q) &= \sum_{n=0}^{\infty} b_k(n) q^n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{k} b_m(n)^* \right) q^n = \sum_{n=0}^{\infty} \sum_{m=0}^{k} b_m(n)^* q^{\Sigma\sigma} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{k} \sum_{i=0}^{b_m(n)^*} q^{\Sigma\sigma(i,n,m)} \end{aligned} \tag{7}$$

where $\sigma(i,n,m)$ is the i'th partition of $B_m(n)^*$. Because for each n we will have $b_k(n)$ copies of q. We now consider the the $k+1$ case. So we are now allowed to use copies of $k+1$ to sum to n. So for every $q^{\Sigma\sigma(i,n,m)}$, since we are covering all integers, we now have, $q^{l(k+1)+\Sigma\sigma(i,n,m)} \quad \forall l \in \mathbb{N}_0$. So

$$\begin{aligned} G_{(b_{k+1}(n))}(q) &= \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{k} \sum_{i=0}^{b_m(n)^*} q^{l(k+1)+\Sigma\sigma(i,n,m)} \\ &= \sum_{l=0}^{\infty} q^{l(k+1)} \sum_{n=0}^{\infty} \sum_{m=0}^{k} \sum_{i=0}^{b_m(n)^*} q^{\Sigma\sigma(i,n,m)} \\ &= \sum_{l=0}^{\infty} q^{l(k+1)} G_{(b_k(n))}(q) \\ &= G_{(b_k(n))}(q) \sum_{l=0}^{\infty} (q^{k+1})^l \\ &= \left(\prod_{j=1}^{k} \frac{1}{1-q^j} \right) \frac{1}{1-q^{k+1}} \\ &= \prod_{j=1}^{k+1} \frac{1}{1-q^j} \\ &\implies P(k+1) = True \end{aligned} \tag{8}$$

So by induction we are done. □

If we take $m \to \infty$ then we get the generating function of $p(n)$.

$$G_{(p(n))}(q) = \lim_{m\to\infty} \prod_{j=1}^{m} \frac{1}{1-q^j} = \prod_{j=1}^{\infty} \frac{1}{1-q^j} \tag{9}$$

3 Generating functions of restricted partitions

We now look at restricted partitions meaning a subset of A_n where all σ obey some condition.

Definition 6. *Let*

$$p(n|\ \textit{some condition on}\ \sigma) \tag{10}$$

denote the number of partitions of n which obey the condition.

Our expression for $G_{(p(n))}(q)$ in (11) gives us a lot of information on the structure of the partitions.

$$\begin{aligned}
G_{(p(n))}(q) &= \prod_{j=1}^{\infty}(q^0 + q^j + q^{2j} + ...) \\
&= (q^0 + q^1 + q^2 + ...)(q^0 + q^2 + q^4 + ...)(q^0 + q^3 + q^6 + ...)... \\
&= (q^0 + q^1 + q^{1+1} + ...)(q^0 + q^2 + q^{2+2} + ...)(q^0 + q^3 + q^{3+3} + ...)...
\end{aligned} \tag{11}$$

So for each n, after expanding this formula, the number of q's in the expanded sum with an exponent of n is precisely $p(n)$. We now look at some restricted partitions.

3.1 Restricting the multiplicity of each λ

Consider the following restricted partitions.

$$p(n|\ \mathrm{mult}(\lambda) = M) \tag{12}$$

Each integer in the partition must appear exactly M times or not appear at all. We are therefore restricted to

$$\begin{aligned}
G_{(p(n|\ \mathrm{mult}(\lambda)=M))}(q) &= (q^0 + q^{1\cdot M})(q^0 + q^{2\cdot M})(q^0 + q^{3\cdot M})... \\
&= \prod_{j=1}^{\infty}(1 + q^{jM})
\end{aligned} \tag{13}$$

Similarly, for

$$p(n|\ \mathrm{mult}(\lambda) \leq M) \tag{14}$$

we get

$$\begin{aligned}
G_{(p(n|\ \mathrm{mult}(\lambda)\leq M))}(q) &= (q^0 + q^1 + q^{1+1} + ... + q^{1\cdot M})(q^0 + q^2 + q^{2+2} + ... + q^{2\cdot M})(q^0 + q^3 + q^{3+3} + ... + q^{3\cdot M})... \\
&= \prod_{j=1}^{\infty}(q^0 + q^j + q^{2j} + ... + q^{j\cdot M}) \\
&= \prod_{j=1}^{\infty}\frac{1-(q^j)^{M+1}}{1-q^j}
\end{aligned} \tag{15}$$

We can generalize this restriction to control the multiplicity of each integer separately through a map, $f : \mathbb{N}_1 \to \mathbb{N}_0 \cup \{\infty\}$. Then

$$G_{(p(n|\ \mathrm{mult}(\lambda)\ \leq\ f(\lambda)))}(q) = \left(\prod_{\substack{j=1\\ f(j)<\infty}}^{\infty}\frac{1-(q^j)^{f(j)+1}}{1-q^j}\right)\left(\prod_{\substack{j=1\\ f(j)=\infty}}^{\infty}\frac{1}{1-q^j}\right) \tag{16}$$

3.2 Restricting the λ's in the partitions

Next we next look at restricting the allowed λ's in the partition to a set $S \subseteq \mathbb{N}_0$. From our expansion of $G_{(p(n))}(q)$, each $\frac{1}{1-q^j}$ term means that all partitions with any number of $\lambda = j$ will be included. So to make a λ not appear in the partition, we simply remove the corresponding $\frac{1}{1-q^\lambda}$ term. So

$$G_{(p(n|\lambda\in S))}(q) = \prod_{j\in S}\frac{1}{1-q^j} \tag{17}$$

Example 1. *If S is a residue class $[a]_b$, for $a, b \in \mathbb{N}$ then*

$$G_{(p(n|\lambda\in[a]_b))}(q) = \prod_{\substack{j=1\\ j\in[a]_b}}^{\infty} \frac{1}{1-q^j} = \prod_{j=0}^{\infty} \frac{1}{1-q^{bj+a}} \tag{18}$$

We can now express $G_{(b_m(n))}(q)$ differently.

Definition 7. *Let $S_m := \{a \in \mathbb{N}_0 : 1 \le a \le m\}$.*

Then

$$G_{(b_m(n))}(q) = G_{(p(n|\lambda\in S_m))}(q) = \prod_{j=1}^{m} \frac{1}{1-q^j} \tag{19}$$

3.3 Restricting the total number of λ's in the partitions

We now look at

$$p(n|\ |\sigma| = N) \tag{20}$$

We can notice a pattern by first looking at $G_{(p(n|\,\mathrm{mult}(\lambda)=1,\,\lambda\in S_3))}(q)$. From (15) we get

$$\begin{aligned} G_{(p(n|\ \mathrm{mult}(\lambda)=1,\,\lambda\in S_3))}(q) &= \prod_{j=1}^{3}(1+q^j) \\ &= 1 + q^1 + +q^2 + q^3 + q^{1+2} + q^{1+3} + q^{2+3} + q^{1+2+3} \end{aligned} \tag{21}$$

To track the number of elements in each partition we track the number of times two q^j's were multiplied. We can track this by adding a variable z.

$$\prod_{j=1}^{3}(1+zq^j) = z^0 1 + z^1 q^1 + z^1 q^2 + z^1 q^3 + z^2 q^{1+2} + z^2 q^{1+3} + z^2 q^{2+3} + z^3 q^{1+2+3} \tag{22}$$

Since each exponent of q represents a σ, the exponent of z is $|\sigma|$. Generalizing this we get

$$G_{(p(n|\ \mathrm{mult}(\lambda)=1,\,|\sigma|=m))}(q,z) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} p(n|\ \mathrm{mult}(\lambda)=1,\ |\sigma|=m) z^m q^n = \prod_{j=1}^{\infty}(1+zq^j) \tag{23}$$

Where this generating function has two variables, z and q. We can generalize this to the non restricted partition. We get that

$$G_{(p(n,\,|\sigma|=m))}(q,z) = \prod_{j=1}^{\infty} \frac{1}{1-zq^j} \quad . \tag{24}$$

where we only consider the terms with a factor of z^m.

3.4 Combining the restrictions

Finally, we can combine these different restrictions. For example

$$G_{(p(n)|\ \mathrm{mult}(\lambda)\le f(\lambda),\,\lambda\in S,|\sigma|=m))}(q,z) = \left(\prod_{\substack{j\in S\\ f(j)<\infty}} \frac{1-(zq^j)^{f(j)+1}}{1-zq^j}\right)\left(\prod_{\substack{j\in S\\ f(j)=\infty}} \frac{1}{1-zq^j}\right) \tag{25}$$

4 Partition identities

We can notice that the generating functions of some restricted partitions are the same. In these cases the number of partitions for each n must be the same. Here are some examples.

Proposition 4.1. $p(n|\, \lambda \in [1]_2) = p(n|\, mult(\lambda) = 1)$. *This is known as Euler's identity.*

Proof.

$$\begin{aligned} G_{(p(n|\,\mathrm{mult}(\lambda)=1))}(q) &= \prod_{j=1}^{\infty}(1+q^j) = \prod_{j=1}^{\infty}\frac{1-q^{2j}}{1-q^j} \\ &= \frac{1-q^2}{1-q}\frac{1-q^4}{1-q^2}\frac{1-q^6}{1-q^3}\frac{1-q^8}{1-q^4}... \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)...} \\ &= \prod_{j=1}^{\infty}\frac{1}{1-q^{2j-1}} \\ &= G_{(p(n|\,\lambda\in[1]_2))}(q) \end{aligned} \tag{26}$$

□

A closely related and more general partition identity would be the following.

Proposition 4.2. $p(n|\, mult(\lambda) < k) = p(n|\, \lambda \notin [0]_k)$

Proof.

$$\begin{aligned} G_{(p(n|\,\mathrm{mult}(\lambda)<k))}(q) &= \prod_{j=1}^{\infty}\frac{1-(q^j)^k}{1-q^j} \\ &= \frac{1-q^k}{1-q}\frac{1-q^{2k}}{1-q^2}\frac{1-q^{3k}}{1-q^3}... \\ &= \prod_{\substack{j=1 \\ j\notin[0]_k}}\frac{1}{1-q^j} \\ &= G_{(p(n|\,\lambda\notin[0]_k)}(q) \end{aligned} \tag{27}$$

□

Proposition 4.3. $p(n|\, mult(\lambda) < \lambda) = p(n|\, \lambda$ *is not a perfect square*$)$

Proof. From our expression for the generating function of $p(n|\ \mathrm{mult}(\lambda) \leq M)$ we get that

$$\begin{aligned}
G_{(p(n|\ \mathrm{mult}(\lambda)<\lambda))}(q) &= \prod_{j=1}^{\infty} \frac{1-(q^j)^j}{1-q^j} \\
&= \frac{1-q^1}{1-q}\frac{1-q^4}{1-q^2}\frac{1-q^9}{1-q^3}... \\
&= \frac{1}{(1-q^2)(1-q^3)(1-q^5)(1-q^6)...} \\
&= \prod_{\substack{j=1 \\ j \text{ not a perfect square}}}^{\infty} \frac{1}{1-q^j} \\
&= G_{(p(n|\ \lambda \text{ is not a perfect square}))}(q) \qquad (28)
\end{aligned}$$

□

Proposition 4.4. *Let $N \subseteq \mathbb{N}$ such that $\forall\, a \in N,\ 2a \notin N$. Let $M := \{2^k a \,|\, \forall a \in N, \forall k \in \mathbb{N}_0\}$ then*

$$p(n|\ \lambda \in N) = p(n|\ \lambda \in M,\ mult(\lambda) = 1) \qquad (29)$$

Proof.

$$\begin{aligned}
G_{(p(n|\ \lambda\in M,\ \mathrm{mult}(\lambda)=1))}(q) &= \prod_{\substack{j=1 \\ j\in M}}^{\infty} (1+q^j) \\
&= \prod_{a\in N}\prod_{k=0}^{\infty}(1+q^{2^k a}) \\
&= \prod_{a\in N}\prod_{k=0}^{\infty}\frac{1-q^{2^{k+1}a}}{1-q^{2^k a}} \\
&= \prod_{a\in N}\frac{1-q^{2^1 a}}{1-q^{2^0 a}}\frac{1-q^{2^2 a}}{1-q^{2^1 a}}\frac{1-q^{2^3 a}}{1-q^{2^2 a}}... \\
&= \prod_{a\in N}\frac{1}{1-q^a} \\
&= G_{(p(n|\ \lambda\in N))}(q) \qquad (30)
\end{aligned}$$

□

5 Expression for $p(n)$ in a few restricted cases

5.1 $p(n|\lambda \in S_1)$

Clearly, $\forall n$ the only option is 1+1+...+1 n times so $p(n|\lambda \in S_1) = 1$.

5.2 $p(n|\lambda \in S_2)$

Here mult(2) uniquely determines the partitions as for every different multiplicity of 2 we have a new partitions. We have a partition for mult(2) $= 0$, mult(2) $= 1$, mult(2) $= 2, ...,$ mult(2) $= \lfloor \frac{n}{2} \rfloor$. So

$$p(n|\lambda \in S_2) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \qquad (31)$$

5.3 $p(n|\lambda \in S_3)$

Here things get more complicated as a combination of 2's and 3's determine the partitions. We therefore try to expand the generating function to get it in the following form

$$G_{(p(n|\lambda \in S_3))}(q) = \frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3} = \sum_{n=0}^{\infty} c_n q^n \tag{32}$$

So that we can conclude, by the definition of the generating function, that $p(n|\lambda \in S_3) = c_n \, \forall n$. The goal will be to use partial fractions to rewrite this as the sum of terms of the form $(1-q)^{-a}$ or $\frac{1}{1-q^a}$ for some integer a since these can be written as a series in q. We already know that $\frac{1}{1-q^a} = \sum\limits_{k=0}^{\infty} q^{an}$. For the other form, $(1-q)^{-a}$, we find its Taylor series. The Taylor series of a function $f(x)$ around q is

$$f(x)|_q = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} q^k \tag{33}$$

For $f(q) = (1-q)^{-a}$ we have

$$f^{(1)}(q) = a(1-q)^{-a-1} \quad ; \quad f^{(2)}(q) = a(a+1)(1-q)^{-a-2} \quad ; \quad f^{(3)}(q) = a(a+1)(a+2)(1-q)^{-a-3} \tag{34}$$

Continuing the pattern we see that

$$f^{(k)}(q) = a(a+1)(a+2)...(a+k-1)(1-q)^{-a-k} = \frac{(a+k-1)!}{(a-1)!}(1-q)^{-a-k} \tag{35}$$

So the Taylor series about q is then

$$(1-q)^{-a} = \sum_{k=0}^{\infty} \frac{(a+k-1)!}{k!(a-1)!} q^k \tag{36}$$

From Wolfram Alpha, we get equation this first line, equation (37). We then express it as the sum of series.

$$\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3} = \frac{q}{9\left(q^2+q+1\right)} + \frac{1}{4q^2-8q+4} + \frac{2}{9q^2+9q+9} - \frac{1}{6q^3-18q^2+18q-6} + \frac{1}{8q+8} - \frac{17}{72q-72} \tag{37}$$

$$= \frac{\frac{q}{9}+\frac{2}{9}}{q^2+q+1} + \frac{\frac{1}{4}}{(1-q)^2} + \frac{\frac{1}{6}}{(1-q)^3} + \frac{\frac{1}{8}}{q+1} + \frac{\frac{17}{72}}{q-1}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(2+n-1)!}{n!(2-1)!} q^n + \frac{1}{6}\sum_{n=0}^{\infty} \frac{(3+n-1)!}{n!(3-1)!} q^n + \frac{\frac{1}{8}(q-1)+\frac{17}{72}(q+1)}{1-q^2} + \frac{\frac{1}{9}(q+2)(q-1)^2}{1-q^3}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(n+1)q^n + \frac{1}{6}\sum_{n=0}^{\infty}\frac{(1+n)(2+n)}{2}q^n + \frac{\frac{13q}{36}+\frac{1}{9}}{1-q^2} + \frac{\frac{1}{9}\left(q^3-3q+2\right)}{1-q^3}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(n+1)q^n + \frac{1}{6}\sum_{n=0}^{\infty}\frac{(1+n)(2+n)}{2}q^n + \frac{13}{36}\sum_{n=0}^{\infty}q^{2n+1} + \frac{1}{9}\sum_{n=0}^{\infty}q^{2n} + \frac{1}{9}\sum_{n=0}^{\infty}q^{3n+3} - \frac{1}{3}\sum_{n=0}^{\infty}q^{3n+1} + \frac{2}{9}\sum_{n=0}^{\infty}q^{3n}$$

$$= \sum_{n=0}^{\infty} q^n \left(\frac{n+1}{4} + \frac{(n+1)(n+2)}{12} + \frac{13q^{n+1}}{36} + \frac{q^{n+1}}{9} + \frac{q^{2n+3}}{9} + \frac{q^{2n+1}}{-3} + \frac{2q^{2n}}{9}\right)$$

$$= \sum_{n=0}^{\infty} q^n \left(\frac{(n+3)^2}{12} - \frac{1}{3} + \frac{13q^{n+1}}{36} + \frac{q^{n+1}}{9} + \frac{q^{2n+3}}{9} + \frac{q^{2n+1}}{-3} + \frac{2q^{2n}}{9}\right) \tag{38}$$

From here our goal is for the large term in the parenthesis to be some integer that depends on n. We have $|q| < 1$ and $p(n) > 0 \implies 0 \leq q < 1 \implies 0 \leq q^n < 1 \, \forall q, n$.

$$\epsilon(n) := -\frac{1}{3} + \frac{13q^{n+1}}{36} + \frac{q^{n+1}}{9} + \frac{q^{2n+3}}{9} + \frac{q^{2n+1}}{-3} + \frac{2q^{2n}}{9} \leq -\frac{1}{3} + \frac{13}{36} + \frac{1}{9} + \frac{1}{9} - 0 + \frac{2}{9} = \frac{17}{36} \tag{39}$$

We have that $\frac{(n+3)^2}{12} + \epsilon(n) \in \mathbb{N}$ so since $\epsilon(n) \leq \frac{17}{36} < \frac{1}{2}$, we get that $\frac{(n+3)^2}{12} + \epsilon(n) = \left[\frac{(n+3)^2}{12}\right]$ where $[x]$ denotes the closest integer to x. So

$$G_{(p(n|\,\lambda \in S_3))}(q) = \sum_{n=0}^{\infty} \left[\frac{(n+3)^2}{12}\right] q^n$$

$$\iff p(n|\,\lambda \in S_3) = \left[\frac{(n+3)^2}{12}\right] \tag{40}$$

6 Euler's Pentagonal Number Theorem

Here we will prove a relation between partitions and pentagonal numbers with the use of generating functions. The pentagonal numbers are illustrated in Figure 1.

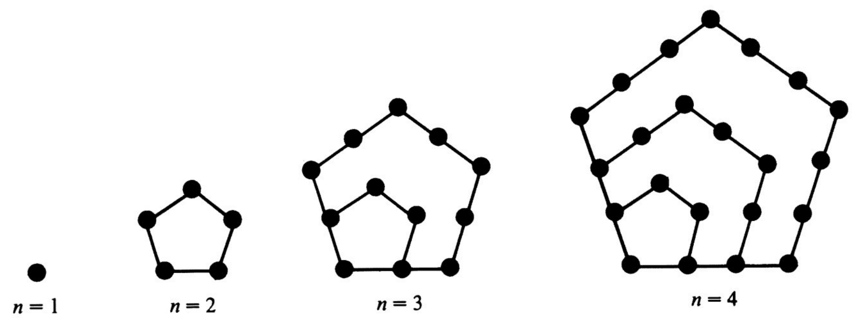

Figure 1: Illustration of the first four pentagonal numbers [2]. They are 1, 5, 12 and 22. In general they are given by $\frac{n(3n-1)}{2}$.

First we define the following function.

Definition 8.

$$f(x, q) := 1 - \sum_{j=1}^{\infty} (1 - xq)(1 - xq^2)(1 - xq^3)...(1 - xq^{j-1})x^{j+1}q^j$$

$$= 1 - \sum_{j=1}^{\infty} \left(\prod_{m=1}^{j-1} (1 - xq^m) \right) x^{j+1} q^j \tag{41}$$

And we also define the finite version.

$$f(x, q)_N := 1 - \sum_{j=1}^{N} \left(\prod_{m=1}^{j-1} (1 - xq^m) \right) x^{j+1} q^j \tag{42}$$

The following proposition will explain how this function is relevant to partitions.

Proposition 6.1.

$$f(1, q)_N = 1 - \sum_{j=1}^{N} \left(\prod_{m=1}^{j-1} (1 - q^m) \right) q^j = \prod_{j=1}^{N} (1 - q^j) \tag{43}$$

Where we recognise the RHS as the denominator for the generating function of $p(n|\,\lambda \in S_N)$.

Proof. We proceed by induction on N. Let

$$P(N) = \text{“} f(1,q)_N = \prod_{j=1}^{N}(1-q^j) \text{”} \tag{44}$$

we prove that $P(N) = True \; \forall N \in \mathbb{N}_1$.
Base case. For $N = 1$ we get on the LHS: $1 - (1-0)q^1 = 1 - q$ and on the RHS we get $1 - q$. So $P(1) = True$.
Induction Hypothesis. Suppose that for some $k \in \mathbb{N}_1$, $P(k) = True$.
Induction step. We get

$$\begin{aligned}
1 - \sum_{j=1}^{k+1}\left(\prod_{m=1}^{j-1}(1-q^m)\right)q^j &= 1 - \sum_{j=1}^{k}\left(\prod_{m=1}^{j-1}(1-q^m)\right)q^j - \left(\prod_{m=1}^{k}(1-q^m)\right)q^{k+1} \\
&= \prod_{j=1}^{k}\left(1-q^j\right) - \left(\prod_{m=1}^{k}(1-q^m)\right)q^{k+1} \\
&= \left(\prod_{j=1}^{k}\left(1-q^j\right)\right)\left(1-q^{k+1}\right) \\
&= \prod_{j=1}^{k+1}\left(1-q^j\right) \\
&\implies P(k+1) = True
\end{aligned} \tag{45}$$

So by induction we are done. □

Taking $N \to \infty$ we get that.

$$f(1,q) = \prod_{j=1}^{\infty}(1-q^j) \tag{46}$$

From here we show that this function has a certain recursive like property.

Proposition 6.2.

$$f(x,q) = 1 - x^2q - x^3q^2f(xq),q) \tag{47}$$

Proof. See [2] for the proof, it involves rearranging $f(x,q)$. □

By iterating $f(x,q)$ we see that

$$\begin{aligned}
f(x,q) &= 1 - x^2q - x^3q^2f(xq),q) \\
&= 1 - x^2q - x^3q^2(1 - x^2q^3 - x^3q^5f(xq^2),q)),q) \\
&\dots \\
&= 1 + \sum_{j=1}^{N-1}(-1)^j(x^{3j-1}q^{\frac{j(3j-1)}{2}} + x^{3j}q^{\frac{j(3j+1)}{2}}) \\
&\quad + (-1)^N\left(x^{3N-1}q^{\frac{N(3N-1)}{2}} + x^{3N}q^{\frac{N(3N+1)}{2}}f(xq^N,q)\right)
\end{aligned} \tag{48}$$

We can prove that the above expression is true.

Proof. We proceed by induction on N. Let

$$P(N) = \text{``}f(x,q) = 1 + \sum_{j=1}^{N-1}\Big((-1)^j(x^{3j-1}q^{\frac{j(3j-1)}{2}} + x^{3j}q^{\frac{j(3j+1)}{2}})\Big) + (-1)^N x^{3N-1}q^{\frac{N(3N-1)}{2}} + (-1)^N x^{3N}q^{\frac{N(3N+1)}{2}} f(xq^N,q))\text{''} \tag{49}$$

we prove that $P(N) = True\ \forall N \in \mathbb{N}_1$.
Base case. For $N = 1$ we get. $f(x,q) = 1 - x^2q - x^3q^2 f(xq,q)$ which is true. So $P(1) = True$.
Induction Hypothesis. Suppose for some $k \in \mathbb{N}_1$, $P(k) = True$.
Induction step. We get

$$\begin{aligned}
f(x,q) &= 1 + \sum_{j=1}^{k-1}(-1)^j(x^{3j-1}q^{\frac{j(3j-1)}{2}} + x^{3j}q^{\frac{j(3j+1)}{2}}) \\
&\quad + (-1)^k x^{3k-1}q^{\frac{k(3k-1)}{2}} + (-1)^k x^{3k}q^{\frac{k(3k+1)}{2}} f(xq^k,q) \\
&= 1 + \sum_{j=1}^{k-1}(-1)^j(x^{3j-1}q^{\frac{j(3j-1)}{2}} + x^{3j}q^{\frac{j(3j+1)}{2}}) \\
&\quad + (-1)^k x^{3k-1}q^{\frac{k(3k-1)}{2}} + (-1)^k x^{3k}q^{\frac{k(3k+1)}{2}} f(xq^k,q) \\
&\quad - (-1)^k x^{3k+2}q^{\frac{3k^2+k+2)}{2}} - (-1)^k x^{3k+3}q^{\frac{3k^2+k+4}{2}} f(xq^{k+1},q) \\
&= 1 + \sum_{j=1}^{k}(-1)^j(x^{3j-1}q^{\frac{j(3j-1)}{2}} + x^{3j}q^{\frac{j(3j+1)}{2}}) \\
&\quad + (-1)^{k+1} x^{3k+2}q^{\frac{3k^2+k+2}{2}} + (-1)^{k+1} x^{3k+3}q^{\frac{3k^2+k+4}{2}} f(xq^{k+1},q) \\
&\implies P(k+1) = True
\end{aligned} \tag{50}$$

So by induction we are done. □

Taking $N \to \infty$ we get

$$f(x,q) = 1 + \sum_{j=1}^{\infty}(-1)^j(x^{3j-1}q^{\frac{j(3j-1)}{2}} + x^{3j}q^{\frac{j(3j+1)}{2}}) \tag{51}$$

Since the other terms go to zero. We recognize the pentagonal numbers in the exponent of q. Now, for $x = 1$ we get

$$f(1,q) = 1 + \sum_{j=1}^{\infty}(-1)^j(q^{\frac{j(3j-1)}{2}} + q^{\frac{j(3j+1)}{2}}) := \sum_{j=0}^{\infty} a_j q^j \quad \text{for } a_j = \begin{cases} (-1)^j & \frac{j(3j\pm1)}{2} \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

Using the generating function of two variables in equation (25) in combination with the generating with the generating function for $\operatorname{mult}(\lambda) = 1$ in equation (14) we have that

$$G_{(p(n,|\sigma|=m|\operatorname{mult}(\lambda)=1))}(q,z) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} p(n,|\sigma|=m|\operatorname{mult}(\lambda)=1)z^m q^j = \prod_{j=1}^{\infty}(1+zq^j) \tag{53}$$

So we see that for $z = -1$ we get our expression in equation (51). So m *even* $\implies z^m = 1$ and

m *odd* $\implies z^m = -1$. By splitting the sum accordingly we get

$$\begin{aligned}
G_{(p(n,|\sigma|=m|\,\mathrm{mult}(\lambda)=1))}(q,z) &= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} p(n|\ |\sigma|=m,\ \mathrm{mult}(\lambda)=1)(-1)^m q^j \\
&= \sum_{j=0}^{\infty} p(n|\ |\sigma|\ \textit{is even},\ \mathrm{mult}(\lambda)=1)q^j - \sum_{j=0}^{\infty} p(n|\ |\sigma|\ \textit{is odd},\ \mathrm{mult}(\lambda)=1)q^j \\
&= G_{(p(n|\ |\sigma|\ \textit{is even},\ \mathrm{mult}(\lambda)=1))}(q) - G_{(p(n|\ |\sigma|\ \textit{is odd},\ \mathrm{mult}(\lambda)=1))}(q) \\
&= \sum_{j=0}^{\infty} a_j q^j \quad \text{for } a_j = \begin{cases} (-1)^j & \frac{j(3j\pm1)}{2} \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{54}$$

So by definition of the generating function

$$p(n|\ |\sigma|\ \textit{is even},\ \mathrm{mult}(\lambda)=1) - p(n|\ |\sigma|\ \textit{is odd},\ \mathrm{mult}(\lambda)=1) = \begin{cases} (-1)^j & \frac{j(3j\pm1)}{2} \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

Which is known as the Euler's pentagonal number theorem. This gives us a recursive way for calculating $p(n)$ since we have

$$G_{(p(n))}(q) = \sum_{j=1}^{\infty} p(n)q^j = \prod_{j=1}^{\infty} \frac{1}{1-q^j} \implies \prod_{j=1}^{\infty}(1-q^j)\cdot\sum_{j=1}^{\infty} p(n)q^j = 1 \tag{56}$$

So from equation (52) we get

$$\left(1+\sum_{j=1}^{\infty}(-1)^j(q^{\frac{j(3j-1)}{2}}+q^{\frac{j(3j+1)}{2}})\right)\sum_{j=1}^{\infty} p(n)q^j = 1 \tag{57}$$

Which gives

$$\begin{aligned}
&p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-... \\
&\quad +(-1)^j p(n-\frac{j(3j-1)}{2})+(-1)^j p(n-\frac{j(3j+1)}{2}) \\
&+ ... = 0
\end{aligned} \tag{58}$$

Using a dynamic programming approach meaning that we store previous values of $p(n)$ as we calculate, we see that this formula for $p(n)$ has a time complexity of just $O(n^{3/2})$. This algorithm was coded in Python and can be found here [1].

7 The Congruence classes of $p(n)$

We shift our interest to $p(n)$ mod(k) for some integer k. For the first 10 primes we calculate which congruence class each $p(n)$ falls into for $1 \leq n \leq 1000$. These 10 plots, one for each prime, are seen below in Figure 2. Let $p_b := \{0 \leq n \leq 10000 \mid p(n) \in [b]_p\}$. We can see from the plots that for the primes, $p = 5$, 7, 11 , there are more elements in p_0 than for the other p_b. These plots don't give any information on which values of n, from 1 to 10000, these seemingly extra values in p_0 come from. But by reducing the number of samples from 10000 to 100 or by taking a random sample of 1000 out of 10000 we get the similar plots (see [1]). This would indicate that these extra values of $n \in p_0$ are distributed evenly from 1 to 10000. Next we can look at how large $|p_0| - |p_b|$ is for $b \neq 0$. We look at how large this deviation is from the expected value of p_0 if the distribution between congruence classes was totally random. In this case we would expect $10000/5 = 2000$, $10000/7 \approx 1428$, and

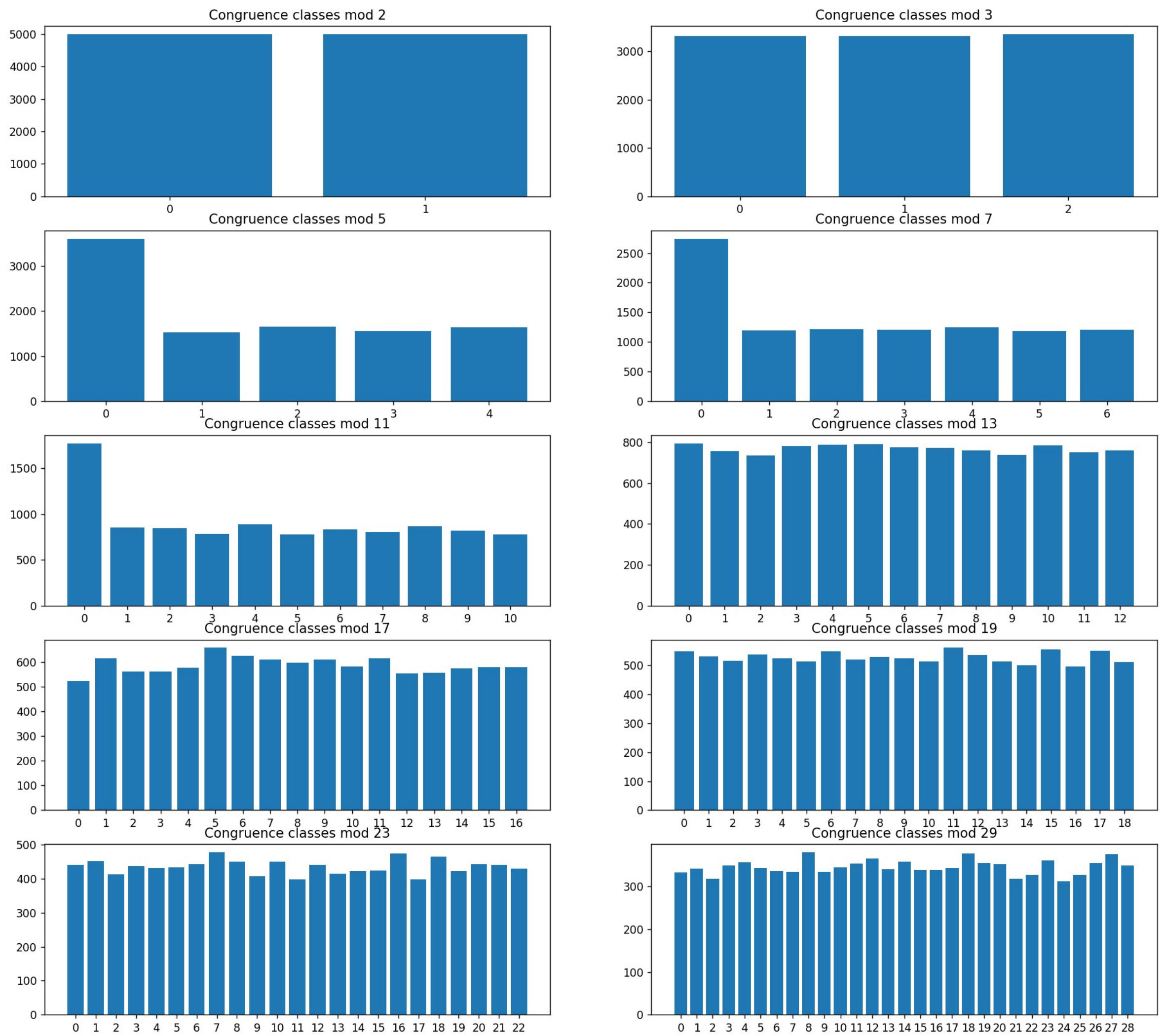

Figure 2: Bar plot of the number of elements of $p(n)$ for $1 \leq n \leq 10000$ in each congruence class mod p. Where p is one of the first 10 primes. Each plot is for a different prime p.

$10000/11 \approx 910$ elements in each p_b for $p = 5,\ 7,\ 11$. We notice that

$$|5_0| = 3611 \approx 3600 = 2000 + \frac{10000 - 2000}{5}$$
$$|7_0| = 2744 \approx 2652 \approx 1428 + \frac{10000 - 1428}{7}$$
$$|11_0| = 1773 \approx 1736 \approx 910 + \frac{10000 - 910}{11}$$

This indicates that for $p = 5, 7, 11$, $|p_0| = \frac{10000}{p} + \Delta p$ where Δp is approximately the average amount of additional elements in p_0 if the other $10000 - 10000/p$ elements were distributed evenly across the $p-1$ congruence classes. If we assume that the extra values of n are evenly distributed from 1 to 10000, this would indicate that when n is in a certain congruence class mod p, $p(n)$ is always congruent to 0 mod p. We therefore make the following conjecture.

Conjecture 1. $\exists\ k_5,\ k_7,\ k_{11} \in \mathbb{N}$ *such that*

$$p(n \mid 5n + k_5) \equiv 0 \ mod\, 5$$
$$p(n \mid 7n + k_7) \equiv 0 \ mod\, 7$$
$$p(n \mid 11n + k_{11}) \equiv 0 \ mod\, 11$$

These end up being true for $k_5 = 4$, $k_7 = 5$, $k_{11} = 6$ and was first conjectured by Ramanujan. A proof is given in [3] p81-87.

References

[1] https://github.com/AvramSilb/IntegerPartitions.

[2] George E Andrews. Euler's pentagonal number theorem. *Mathematics Magazine*, 56(5):279–284, 1983.

[3] George E Andrews and Kimmo Eriksson. *Integer partitions*. Cambridge University Press, 2004.

[4] Lokenath Debnath. A brief history of partitions of numbers, partition functions and their modern applications. *International Journal of Mathematical Education in Science and Technology*, 47(3):329–355, 2016.