MCGILL UNIVERSITY

MATH377

HONOURS NUMBER THEORY

Farey Diagrams

Author
Bronwyn WALSH

April 12, 2023

1 Introduction - Historical Background

Quadratic forms are perhaps the oldest topic discussed in this paper. One of the earliest integer quadratic forms studied was $x^2 - ny^2 = c$. As far back as 682 BCE, the mathematician Brahmagupta studied solutions to this quadratic form. In the late 18th century, Lagrange developed the particular relation between Pell's equation, $x^2 - ny^2 = 1$, and continued fractions. [Lag67]

The use of continued fractions is particularly important for applications of the Farey topograph, which are a geometric interpretation of quadratic forms. The Farey topograph is a comparatively niche subject, however the basis of its function has been written about by Allen Hatcher. [Hat22]

The topograph can be viewed as an application of Farey diagrams which are themselves an extrapolation of the Farey Sequence/Series. Although the sequence is named after 19th century geologist John Farey, Sr., Charles Haros had first proved the rule by which the sequence expands ten years prior to Farey's work. [Wri60]

2 Mathematical Background

Definition 1 (Quadratic Form)**.** *A quadratic form $Q(x, y)$ is a polynomial over $K[x]$, K a commutative ring, such that every term is of degree two.*

e.x. $Q(x, y) = ax^2 + bxy + cy^2$; $a, b, c \in K$

Definition 2 (Farey Sequence)**.** *The Farey sequence is a sequence of fully reduced rationals between* 0 *and* 1

*The rule by which we obtain adjacent terms in the sequence is called the **mediant operation**, and is defined as* $\frac{a_{n+1}}{b_{n+1}} = \frac{a_{n-1}+a_n}{b_{n-1}+b_n}$.

$$
\begin{aligned}
F_1 &= \{\frac{0}{1}, \frac{1}{1}\} \\
F_2 &= \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\} \\
F_3 &= \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\} \\
F_4 &= \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\} \\
&\vdots
\end{aligned}
$$

The negative Farey sequence consists of negative rationals and is constructed by the operation $\frac{a_{n+1}}{b_{n+1}} = \frac{-|a_{n-1}|-|a_n|}{|b_{n-1}|+|b_n|}$

$$
\begin{aligned}
F_1 &= \{\frac{0}{1}, \frac{-1}{1}\} \\
F_2 &= \{\frac{0}{1}, \frac{-1}{2}, \frac{-1}{1}\} \\
F_3 &= \{\frac{0}{1}, \frac{-1}{3}, \frac{-1}{2}, \frac{-2}{3}, \frac{-1}{1}\} \\
F_4 &= \{\frac{0}{1}, \frac{-1}{4}, \frac{-1}{3}, \frac{-2}{5}, \frac{-1}{2}, \frac{-3}{5}, \frac{-2}{3}, \frac{-3}{4}, \frac{-1}{1}\} \\
&\vdots
\end{aligned}
$$

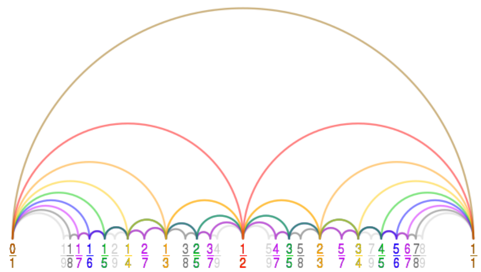
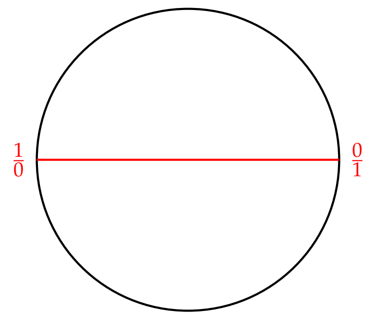


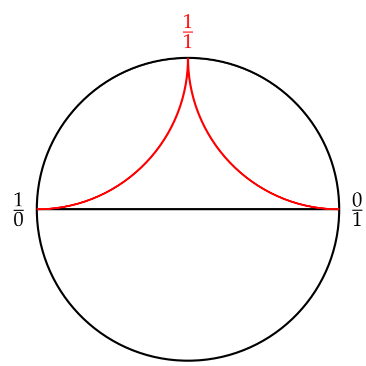
Figure 1: Finite representation of Farey sequence [Cmg]

Definition 3 (Farey Diagram)**.** *The Farey diagram is an illustration which follows from the Farey sequence. It is drawn as follows.*

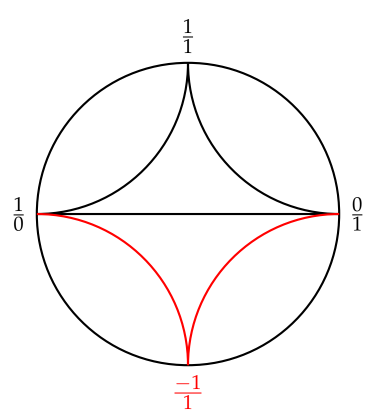
First, a circle is drawn with the vertices $\frac{0}{1}$ *and* $\frac{1}{0}$ *(infinity) labelled, and a path drawn between the two.*



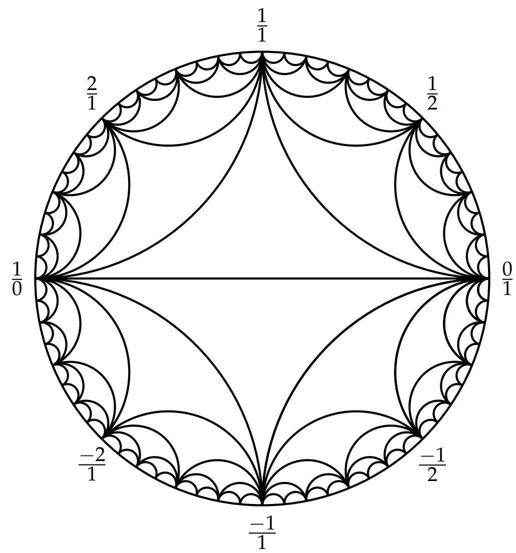
Next the mediant vertex is added at the midpoint along the circle, and a path of curvilinear triangles is drawn between it and its 'parent' vertices.



Continuing in this fashion, we can construct the entire upper half of the diagram. The lower half of the diagram is constructed similarly using the negative mediant operation.



The Farey Diagram can only ever be drawn finitely, however the following illustration shows a representation of the diagram at a later stage.



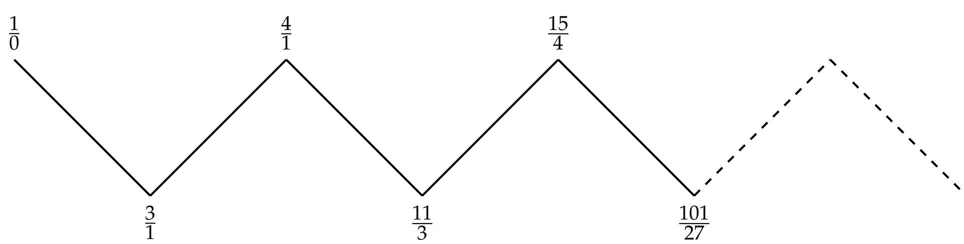
Definition 4 (Farey Triangle Strip)**.** *Let $n \in \mathbb{N}$. Let n be nonsquare*. Consider the continued fraction for $\sqrt{n}$:*

$$n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

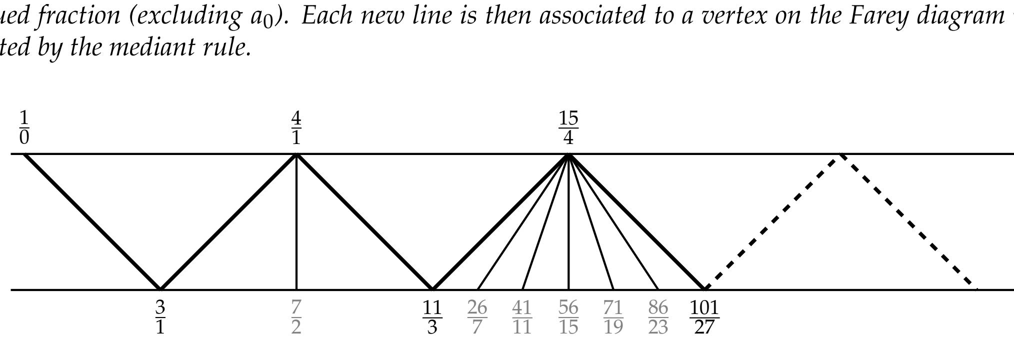
The Farey triangle strip is constructed as follows.

First consider the zig-zag line with the first vertex $:= \frac{1}{0}$, second vertex $:= \frac{a_0}{1}$, and successive vertices labelled as the consecutive convergents of the continued fraction of $\sqrt{n}$.

e.x. $\sqrt{14} = 3 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{\ddots}}}}}$



Each major triangle is then subdivided into a number of minor regions determined by consecutive values in the continued fraction (excluding a_0). Each new line is then associated to a vertex on the Farey diagram which is calculated by the mediant rule.



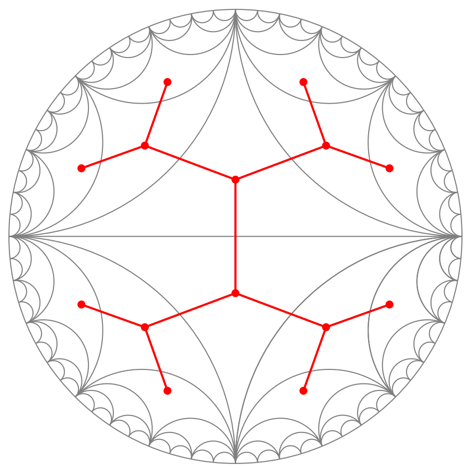
This pictorial representation is simply an adapted drawing of a particular path between edges on the Farey diagram, where the same vertices are connected in both representations. Indeed, in the example of $\sqrt{14}$, the minor triangles create a path via successive applications of the mediant operation such that $\frac{15+86}{4+23} = \frac{101}{27}$, which is the next convergent of $\sqrt{14}$.

**Note: if n square, then $\sqrt{n} \in \mathbb{N}$, hence the continued fraction of $\sqrt{n}$ is finite.*

Definition 5 (the Topograph)**.** *The topograph of the Farey diagram is a mapping of values $Q(a,b) = m$ of a quadratic form $Q(x,y)$. Like the Farey Diagram, we can only draw finite portions of the topograph, however it fractalizes infinitely.*

It is constructed as follows.

1. *A vertex is drawn in each curvilinear triangle of the Farey Diagram.*
2. *An edge is drawn between every pair of vertices which are seperated by a line on the Farey diagram.*



3. *Every region touches one fraction $\frac{a}{b}$ on the outside of the Farey Diagram [Hat22], and so we associate the value $Q(a,b) = m$ to this region.*

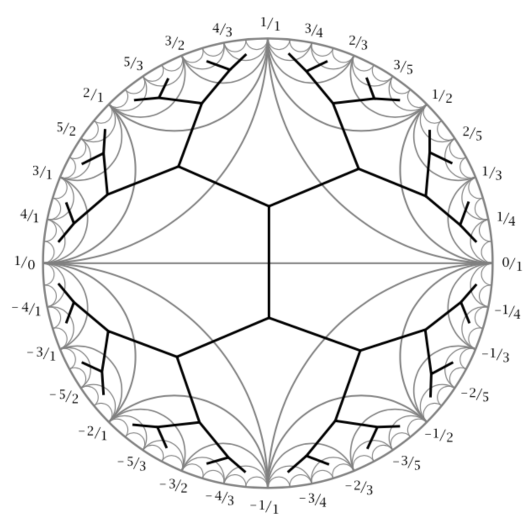
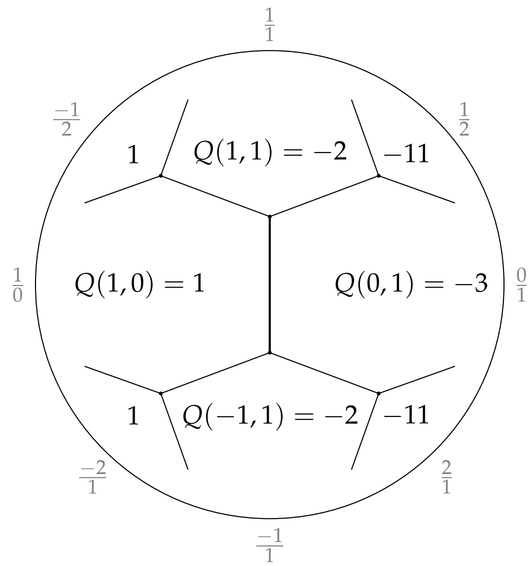


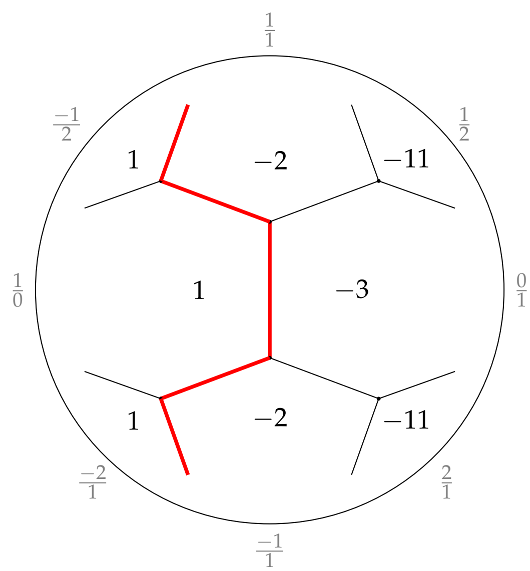
Figure 2: Finite illustration of the topograph [Hat22]

e.x. *Consider the quadratic form* $Q(x,y) = x^2 - 3y^2$

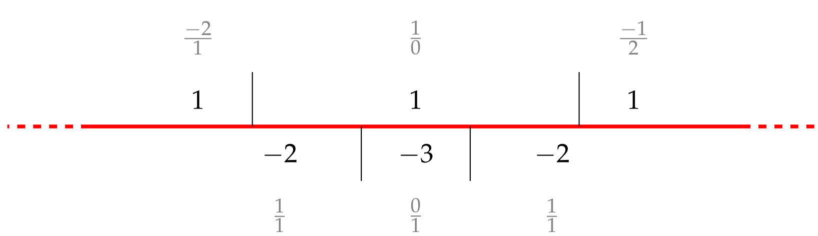


Definition 6 (the River). *By the monotonicity property (Lemma 2.2), positive regions of the topograph tend to $+\infty$ and negative regions of the topograph tend to $-\infty$. The separator line- or the river- is the path that follows the border between positive and negative regions.*

Consider the quadratic form $Q(x,y) = x^2 - 3y^2$

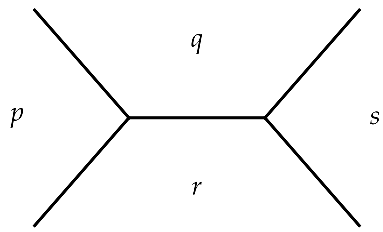


In order to better visualise the river, we often use the following 'flattened' depiction.



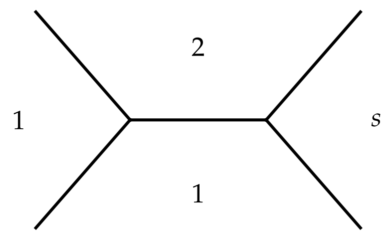
Theorem 1 (Arithmetic Progression of the Topograph)**.** *Values of the topograph can be determined from the following arithmetic progression:*

Given three regions surrounding a vertex in the topograph with values p, q, r:



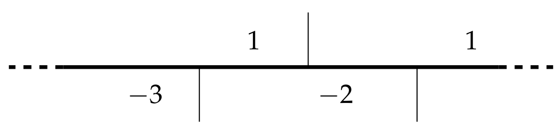
the value of the fourth region s follows the arithmetic progression $p, q + r, s$. [Hat22]

e.x.



$s = 5$ *since the arithmetic progression is* $1, 1 + 2, s$

The arithmetic progression rule also allows us to generate the entire river based on three initial values



Here $q = -3$, $p = 1$, $r = -2 \implies s = 1$

Theorem 2 (Periodicity of the River)**.** *The river is periodic for quadratic forms $Q(x, y) = x^2 - ny^2$ where $n \in \mathbb{N}$ is non-square. [Hat22]*

Lemma 2.1 (the Discriminant)**.** *Let h be the arithmetic increment between a set of regions surrounding a common vertex. Consider regions p, q, r, s which follow the arithmetic progression $p, q + r, s$*
i.e. $h = (q + r) - p = s - (q + r)$
Call the following edge h

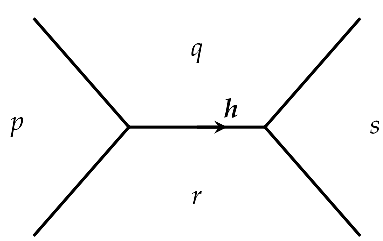


Figure 3: [Hat22]

If an edge in the topograph of a form $Q(x, y)$ is labeled h with adjacent regions labeled p and q, then the discriminant of $Q(x, y)$ is $h^2 - pq$. [Hat22]

Lemma 2.2 (Monotonicity Property). *Let h be an edge (as defined in previous lemma) between two regions q, r. If q, r, h all positive, then so are $s := q + r + h$ and the edges between q, s and r, s. Also, $s > q, r$. [Hat22] For the sake of brevity, the monotonicity property is not proved in this paper.*

3 Proofs

Theorem 1. (Proof adapted from Topology of Numbers, pg. 83 [Hat22])

Let $Q(x, y) = x^2 - ny^2$ be a quadratic form with $n \in \mathbb{N}$ s.t. n non-square (WLOG).

Consider the following set of edges in the topograph, where $p = Q(x_2 - x_1, y_2 - y_1)$, $q = Q(x_1, y_1)$, $r = Q(x_2, y_2)$

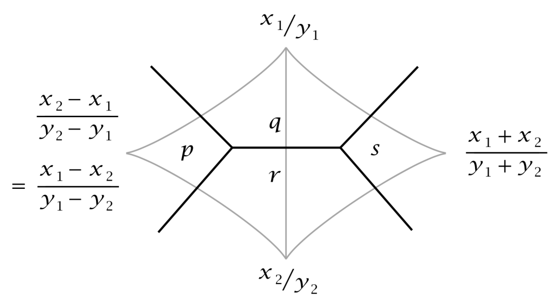


Figure 4: [Hat22]

By the mediant operation, the vertex corresponding to the region s is $\frac{x_1+x_2}{y_1+y_2}$, since the parent vertuces are $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$.

$$
\begin{aligned}
s = Q(x_1 + x_2, y_1 + y_2) &= (x_1 + x_2)^2 + n(y_1 + y_2)^2 \\
&= {x_1}^2 + 2x_1x_2 + {x_2}^2 + n{y_1}^2 + 2ny_1y_2 + n{y_2}^2 \\
&= ({x_1}^2 + n{y_1}^2) + ({x_2}^2 + n{y_2}^2) + 2x_1x_2 + 2ny_1y_2 \\
&= q + r + (2x_1x_2 + 2ny_1y_2)
\end{aligned}
$$

and

$$
\begin{aligned}
p = Q(x_1 - x_2, y_1 - y_2) &= (x_1 - x_2)^2 + n(y_1 - y_2)^2 \\
&= {x_1}^2 - 2x_1x_2 + {x_2}^2 + n{y_1}^2 - 2ny_1y_2 + n{y_2}^2 \\
&= ({x_1}^2 + n{y_1}^2) + ({x_2}^2 + n{y_2}^2) - 2x_1x_2 - 2ny_1y_2 \\
&= q + r - (2x_1x_2 + 2ny_1y_2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
p + s &= q + r + (2x_1x_2 + 2ny_1y_2) + q + r - (2x_1x_2 + 2ny_1y_2) \\
p + s &= q + r + q + r \\
&\implies (q + r) - p = s - (q + r) \\
&\implies p, q + r, s \text{ is an arithmetic progression}
\end{aligned}
$$

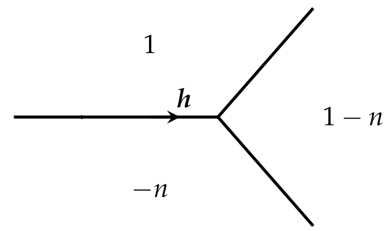
□

Lemma 2.1. (Proof adapted from Topology of Numbers, pg. 104 [Hat22])

Let $Q(x,y) = x^2 - ny^2$ be a quadratic form with $n \in \mathbb{N}$ s.t. n non-square (WLOG).

Base Case:

$Q(0,1) = -n$, $Q(1,0) = 1$ are the $\frac{0}{1}$ and $\frac{1}{0}$ regions of the topograph. Let h be the edge pointing to $\frac{1}{1}$ which separates the two regions.

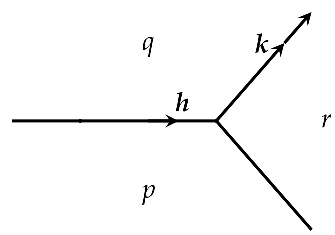


Since the $\frac{1}{1}$ region of the topograph is $Q(1,1) = 1 - n$, $h = 0$.

The discriminant of $Q = 4n = h^2 - 4Q(1,0)Q(0,1)$, hence for the base case the lemma holds.

General Case:

Suppose now that there exist two edges h (which separates regions p and q), and k (which separates regions q and r).



Suppose $\triangle Q = h^2 - 4pq$.
By definition, $r = p + q + h$ and $k = q + r - p = h + 2q$

$$\begin{aligned}
\implies k^2 - 4rq &= h^2 + 4hq + 4q^2 - 4rq \\
&= h^2 + 4hq + 4q^2 - 4(p + q + h)q \\
&= h^2 - 4pq + (4hq + 4q^2 - 4hq - 4q^2) \\
&= h^2 - 4pq \\
&= \triangle Q
\end{aligned}$$

Hence by induction, the hypothesis is true.

□

Theorem 2. (Proof adapted from Topology of Numbers, pg. 106 [Hat22])

Let $Q(x,y) = x^2 - ny^2$ a quadratic form, with $n \in \mathbb{N}$ non-square.

Since every region of the topograph is non-zero, each region must be strictly positive or negative. Hence, there must be a collection of edges which separate the positive and negative regions.

Consider such a separating edge

the value of Q in the next region must also be either positive or negative, thus one of the two edges branching from the initial separating edge must itself be a separating edge. By repeating this argument we can build out a separating line such that it extends infinitely in two directions from our

Figure 5: [Hat22]

initial edge. The edges which branch away from this line are notably not separating edges by the monotonicty property (Lemma 2.2).

As a result, the separating line is unique, and is referred to as the River.

By Lemma 2.1, if we consider an edge labelled h along the river with adjacent regions p and $-q$ s.t. $p, q > 0$, $\triangle = h^2 + 4pq$ must be the discriminant of $Q(x, y)$.

$\implies \triangle > 0\,, |h|, p, q < \triangle$

$\triangle Q$ constant (WRT Q)

$\implies$ h, p, q have finitely many possible values along the river, thus each edge has finitely many possible combinations of h, p, q.

The river is infinite, hence the values of h, p, q at a single edge must at some point repeat.

By Theorem 1, if we have three regions of the topograph which surround a common vertex, then we can generate the rest of the topograph by arithmetic progression.

Hence given h, p, q (where h is the arithmetic progression increment) we can find the third region r necessary to generate the rest of the topograph.

This implies that since h, p, q repeat at some point on the river, the entire sequence along the river must be periodic, as h, p, q will generate the same values every time they appear on the river. $\square$

4 Original Work

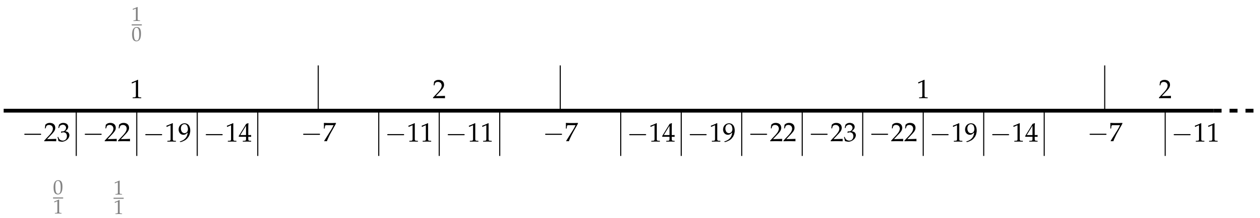
Observation: In *Topology of Numbers*, Hatcher makes the following observation (without formal proof):

The river of a quadratic form $Q(x, y) = x^2 - ny^2$, $n \in \mathbb{N}$ provides the infinite continued fraction of $\sqrt{n}$. [Hat22]

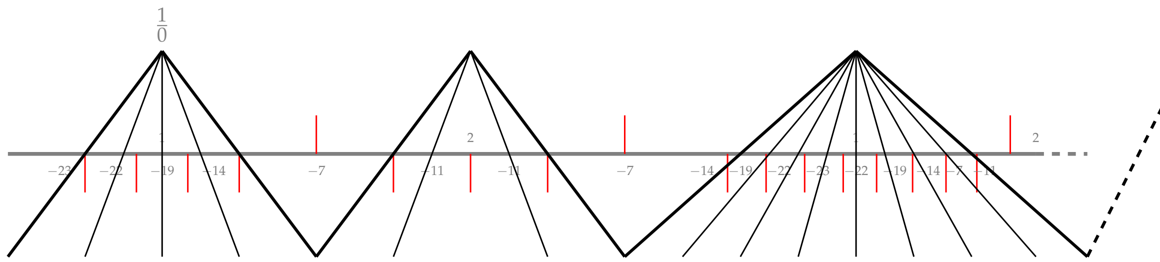
The continued fraction of $\sqrt{n}$ can be computed as follows. Given the river for the quadratic form $Q(x, y) = x^2 - ny^2$, consider the section beginning at the vertices $\frac{1}{0}$, $\frac{0}{1}$, and $\frac{1}{1}$ which extends infinitely to the right.

The number of edges before the pattern switches sign $(\pm)$ corresponds to the next value of the continued fraction.

Consider, for example, $Q(x, y) = x^2 - 23y^2$. The river is as follows



We observe that the upward/downward (red) edges of the river align with each region along the Farey triangle strip for $\sqrt{23}$.



By construction of the Farey triangle strip, the number of minor triangles embedded in each large triangle is in fact the next value in the continued fraction of $\sqrt{n}$. Since each set of upward or downward edges before a switch aligns with the minor regions in each triangle along Farey triangle strip, the continued fraction can be calculated via the method outlined in the observation.

Beginning at the region associated with $\frac{1}{0}$, we first encounter 4 negative edges, then 1 upward edge, then 3 downward edges, then 1 upward edges, then 8 downward edges, at which point the river repeats. This means that the continued fraction of $\sqrt{23}$ is $[4;\overline{1,3,1,8}]$ or $4+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\vdots}}}}}$

The following Java program uses the river of quadratic forms $Q(x,y) = x^2 - ny^2$, where $n \in \mathbb{N}$, (n non-square) to compute the infinite continued fraction of $\sqrt{n}$.
In this program, the sequence after ';' is repeated infinitely.

https://github.com/BronwynWalsh/Continued-Fraction-Calculator

This program implements and uses three methods:

i. $Q(x,n,y)$ which computes the value of $x^2 - ny^2$

ii. $aProgression(a,b)$ which computes the third value c in the arithmetic progression a,b,c

iii. $bigFraction(n)$, which computes the infinite continued fraction of $\sqrt{n}$

Note: all inputs are integers, and bigFraction is assumed to take a positive, non-square integer

The algorithm proceeds as follows:

The first three values which will be used to compute the rest of the river are computed:

$p = Q(1,n,0)$
$q = Q(0,n,1)$
$r = Q(1,n,1)$

The next value is then found using $aProgression(p, q + r)$. This is precisely the arithmetic progression rule applied to the river.

Next the algorithm recursively builds the river. Once the initial p, q, r values come up again, however, the river will repeat by Theorem 2.

At this point the algorithm halts, and analyzes the number of consecutive positive or negative values. These values correspond to the number of successive upward or downward facing edges, which is indeed the infinite continued fraction of $\sqrt{n}$.

Corollary. $\forall n \in \mathbb{N}$ *, the continued fraction of* $\sqrt{n}$ *is periodic.*

Proof. Let $Q(x, y) = x^2 - ny^2$ be a quadratic form with $n \in \mathbb{N}$ s.t. n non-square.

By the observation, the continued fraction of $\sqrt{n}$ can be obtained by the number of edges before a sign switch on the river.

This is a periodic pattern by Theorem 2, hence the continued fraction of $\sqrt{n}$ is periodic. □

5 Further Research and Open Questions

Farey Diagrams are a rather niche subject in a well studied field, as they are a geometric interpretation of quadratic forms and continued fractions. Indeed there are few comprehensive books written on the subject, with *Topology of Numbers* by Allen Hatcher being the main reference.

As a result, much remains to be proved, let alone discovered. A key example is the observation identified previously: the river of the quadratic form $Q(x, y) = x^2 - ny^2$ provides the infinite continued fraction of $\sqrt{n}$ when n is a positive, non-square integer.

This result is not proved in *Topology of Numbers*, (although examples are provided), and while it likely has proofs which use notions such as $SL_2(\mathbb{Z})$ (which relates to the translation and periodicity of the topograph), or other geometric applications of Farey diagrams, these topics unfortunately go beyond the scope of this particular paper. In fact Hatcher states even more generally that any quadratic irrational can be found by this method.

The Farey topograph also provides insight into solutions to Pell's equation. In particular, since every solution to $Q := x^2 - ny^2 = 1$ is a convergent of $\sqrt{n}$ [Mon91] and every convergent appears on the river of Q, every solution must also appear on the river. In other words, every time a 1 appears on the river, the corresponding vertex on the Farey diagram $(\frac{a}{b})$ is a solution. The difficulty arises in finding an efficient method for back-tracking and retrieving the Farey diagram vertex from the river.

References

[Lag67] Joseph Louis Lagrange. *Oeuvres de Lagrange*. Vol. 1. DE M. J.-A. SERRET, 1867, pp. 671–731.

[Wri60] G. H. Hardy; E. M. Wright. *an Introduction to the Theory of Numbers*. 4th. Claredon Press, 1960, p. 36.

[Mon91] Ivan Niven; Herbert S. Zuckerman; Hugh L. Montgomery. *an Introduction to the Theory of Numbers*. 5th. New York: Wiley, 1991, pp. 351–359.

[Hat22] Allen Hatcher. *Topology of Numbers*. AMS, 2022.

[Cmg] Cmglee. *Farey diagram horizontal arc 9.svg*. URL: https://commons.wikimedia.org/wiki/File:Farey_diagram_horizontal_arc_9.svg.

Note: all illustrations without references are either original or public domain