On the Multiplicity of the Ramanujan-τ function

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April, 2023

Abstract

This paper outlines a proof of both the multiplicative property and recurrence relation of the Ramanujan-Tau function. It also outlines a deterministic pattern for the prime-power recurrence with proof. This gives rise to a general formula such that if we have the values of tau for the primes dividing its input, we can determine its value. The Tau function was defined in 1916 by Ramanujan and it maps the natural numbers to the coefficients of the Weierstrass-Delta function, which is a modular form. The function has a number of interesting number theoretic properties, of which its multiplication is the main focus of this paper. In 1916, Ramanujan observed the function to be multiplicative for relatively prime integers, and observed a recurrence relation of prime powers that was proven my Mordell in 1917. Note: this paper assumes elementary knowledge of modular forms and complex analysis.

1 Introduction

In 1916 Ramanujan happened to be exploring the properties of the function:

$$x\prod_{n=1}^{\infty}(1-x^n)^{24} = x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 - 16744x^7 + ... \qquad (1)$$

when he noticed that the coefficients were multiplicative.

For example of we take the coefficients of x^2 and that of x^3 which are -24 and 252 , respectively, then multiplying them gives $(-24)(252) = -6048$ which happens to be the coefficient of x^6.

Ramanujan noticed this pattern held for any two integers if they were relatively prime, and thus decided to define a function, namely the τ function, where $\tau(n)$ was defined to be the value of the coefficient of x^n. [1] The significance of his work was understood by Ramanujan at the time as he was either not aware of the mathematics or they were beyond his time.[4]

It turns out that (1) belongs to a class of functions called Modular forms. Modular forms is a study native to Complex analysis, in which we define a complex function that is somewhat invariant under the actions of the modular group $SL_2(\mathbb{Z})$. The entire modular group can be generated by the 2 transformations corresponding to the function equations:

1. $g(z+1) = g(z)$

2. $g(-\frac{1}{z}) = g^k f(z)$

In essence a modular form of weight k is a complex function g which satisfies the two equations above. (on top of additional complex analytic holomorphic and meromorphic properties) [6]

The significance of the τ function its providing a pathway through which complex analysis and Number theory, fields which were long believed to have little in common, are in fact intertwined. The nature of Ramanujan's work was rooted in his own passion for mathematics. [4] is only fitting given that Ramanujan was a self-taught mathematician living thousands of miles away to the relevant minds sitting atop the upper echelons of academia. As a matter of fact, had it not been for his sending a letter to leading mathematician G.H. Hardy, much of Ramanujan's work would have gone unnoticed. [4]

Ramanujan's work on the τ function has led to major developments on the Langlands program, which is a project that was started in 1967 when Mathematician Robert Langland sent Number theorist André Weil a letter containing a series of conjectures linking harmonic analysis to number theory. [8] Number theoretic properties of $\tau(n)$, which is implicitly defined by a complex function, is what characterizes its significance. Since Ramanujan was alive before this time (died in 1920), he was able to explore the properties of $\tau(n)$ without pigeonholing it into any open problem at the time. In other words, he was simply fascinated by it.

2 Preliminaries

2.1 The Weierstrass-Delta Function

2.1.1 Preliminaries for defining $\Delta(z)$

- Let $\mathbb{H} = \{\omega \in \mathbb{C} : \mathbf{Im}(\omega) > 0\}$ denote the upper half plane
- let $q : \mathbb{H} \longrightarrow \mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ be a holomorphic mapping given by $q(z) = e^{2\pi i z}$, we write $q = q(z)$

Remark 1. *Since* $z = x + iy \in \mathbb{H} \iff y > 0$

$$q(z) = e^{2\pi i z} = e^{2\pi i x} e^{-2\pi y} \in \mathbb{D}$$

2.1.2 Defining $\Delta(z)$:

For $z \in \mathbb{H}$

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \tag{2}$$

2.1.3 Expansion of $\Delta(z)$:

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 + R_{10}(q) \tag{3}$$

where $R_n(q)$ denotes the remainder of the function starting from the n-th coefficient

Elementary properties of $\Delta(z)$

- $\Delta(z)$ is a modular form of weight 12 and satisfies the equations:
  1. $\Delta(z) = \Delta(z+1)$
  2. $\Delta(-\frac{1}{z}) = z^{12}\Delta(z)$
- $\Delta(z)$ is a cusp form, meaning it has no constant term in its Fourier expansion. [6]

2.2 The Ramanujan τ function

The definition of the $\tau : \mathbb{N} \longrightarrow \mathbb{Z}$ function follows implicitly and maps the n-th term to its respective Fourier coefficient.

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} \tag{4}$$

2.3 Properties of the Ramanujan τ function

Ramanujan observed the following about the function but was unable to prove them. They were proven by Mordell in 1917, only one year later. [1] [2]

- **Multiplicative Property:** (Proof in section 3) For any $m, n \in \mathbb{Z}$ such that $(m, n) = 1$ (Proof in [2]
$$\tau(mn) = \tau(m)\tau(n) \tag{5}$$
- **A recurrence relation:** (Proof in section 3) For any prime p and $k \geq 2$ (Proof in [2]
$$\tau(p^k) = \tau(p)\tau(p^{k-1}) - p^{11}\tau(p^{k-2}) \tag{6}$$

2.4 Other important results about the *tau* function

- **Ramanujan's Conjecture** For any prime p, $|\tau(p)| \leq 2p^{\frac{11}{2}}$ [1] (Proof by Deligne in 1974 [4])

3 Proofs

Theorem 2. ***Multiplicative Property:*** *For any $m, n \in \mathbb{Z}$ such that $(m, n) = 1$*

$$\tau(mn) = \tau(m)\tau(n) \tag{7}$$

I will break this theorem down into multiple lemmas that offer insight into the multiplicity of the τ function. This proof was adapted from [7] and [2].

Let Δ denote the modular discriminant function. Pick any prime p and Define:

$$f(z) = p^{12}\Delta(pz) + \sum_{k=0}^{p-1} \Delta\left(\frac{z+k}{p}\right) \tag{8}$$

Lemma 3. *With f as in (8)*

$$f(z+1) = f(z)$$

Proof.

$$f(z+1) = p^{12}\Delta(p(z+1)) + \sum_{k=0}^{p-1} \Delta\left(\frac{(z+1)+k}{p}\right) = p^{12}\Delta(pz+p) + \sum_{k=0}^{p-1} \Delta\left(\frac{(z+1)+k}{p}\right)$$

Since Δ is a cusp form of weight 12, $\Delta(pz+p) = \Delta(pz+p-1) = \Delta(pz+p-2) = ... = \Delta(pz)$

$$\Rightarrow f(z+1) = p^{12}\Delta(pz) + \sum_{k=0}^{p-2} \Delta\left(\frac{z+(k+1)}{p}\right) + \Delta\left(\frac{(z+1)+(p-1)}{p}\right)$$

$$\Rightarrow f(z+1) = p^{12}\Delta(pz) + \sum_{k=1}^{p-1} \Delta\left(\frac{z+k}{p}\right) + \Delta\left(\frac{z+p}{p}\right)$$

Modularity of Δ implies that $\Delta\left(\frac{z+p}{p}\right) = \Delta\left(\frac{z}{p}+1\right) = \Delta\left(\frac{z}{p}\right)$ So,

$$f(z+1) = p^{12}\Delta(pz) + \sum_{k=1}^{p-1} \Delta\left(\frac{z+k}{p}\right) + \Delta\left(\frac{z}{p}\right) = p^{12}\Delta(pz) + \sum_{k=0}^{p-1} \Delta\left(\frac{z+k}{p}\right) = f(z)$$

□

Lemma 4. *With f as in (8)*

$$f(-\frac{1}{z}) = z^{12} f(z)$$

Proof.

$$f\left(-\frac{1}{z}\right) = p^{12}\Delta\left(-\frac{p}{z}\right) + \sum_{k=0}^{p-1} \Delta\left(\frac{(-\frac{1}{z})+k}{p}\right)$$

$$\Rightarrow f\left(-\frac{1}{z}\right) = p^{12}\Delta\left(-\frac{1}{\frac{z}{p}}\right) + \sum_{k=0}^{p-1} \Delta\left(\frac{zk-1}{zp}\right) = p^{12}\left(\frac{z^{12}}{p^{12}}\right)\Delta\left(\frac{z}{p}\right) + \sum_{k=0}^{p-1} \Delta\left(\frac{zk-1}{zp}\right)$$

$$\Rightarrow f\left(-\frac{1}{z}\right) = z^{12}\Delta\left(\frac{z}{p}\right) + \sum_{k=0}^{p-1} \Delta\left(\frac{zk-1}{zp}\right) = z^{12}p^{12}\Delta(z) + \sum_{k=0}^{p-1} \Delta\left(\frac{zk-1}{zp}\right)$$

Now we want to show that:

$$\Delta\left(\frac{zk-1}{zp}\right) = z^{12}\Delta\left(\frac{z+k}{p}\right)$$

Recall that since Δ is a modular form of weight 12 for any $\omega \in \mathbb{H}$, and $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$

$$\Delta\left(\frac{a\omega+b}{c\omega+d}\right) = (c\omega+d)^{12}\Delta(\omega) \tag{9}$$

$$\Delta\left(\frac{zk-1}{zp}\right)=\Delta\left(\frac{a\omega+b}{c\omega+d}\right),\ z=c\omega+d,\ \text{and}\frac{z+k}{p}=\omega$$

We thus aim to show that

$$a\omega+b=\frac{zk-1}{p}\Rightarrow a\left(\frac{z+k}{p}\right)+b=\frac{az+(ak+bp)}{p}=\frac{zk-1}{p}$$

We can make use of the fact that for $1\leq k\leq p-1\Rightarrow gcd(p,k)=1$

$$\Rightarrow \exists m,l\in\mathbb{Z}\text{ such that }lk+mp=1$$

We set $(-l)k+(-m)p=ak+bp=a(-d)+bc=-1$ and that since such integers that satisfy $\Delta\left(\frac{a\omega+b}{c\omega+d}\right)=(c\omega+d)^{12}\Delta\left(\omega\right)=\Delta\left(\frac{zk-1}{zp}\right)=z^{12}\Delta\left(\frac{z+k}{p}\right)$ So,

$$\Rightarrow f\left(-\frac{1}{z}\right)=z^{12}\left(p^{12}\Delta\left(z\right)+\sum_{k=0}^{p-1}\Delta\left(\frac{z+k}{p}\right)\right)=z^{12}f(z)$$

□

Corollary 5. *For any prime p and $k\in\mathbb{Z}^{+}$*

$$\tau(p^k)=\tau(p)\tau(p^{k-1})-p^{11}\tau(p^{k-2})$$

Proof. With f as in (8), define

$$F(z)=\frac{f(z)}{\Delta(z)}$$

It immediately follows that:

1. $F(z+1)=\frac{f(z+1)}{\Delta(z+1)}=\frac{f(z)}{\Delta(z)}=F(z)$
2. $F\left(-\frac{1}{z}\right)=\frac{f\left(-\frac{1}{z}\right)}{\Delta\left(-\frac{1}{z}\right)}=\frac{z^{12}f(z)}{z^{12}\Delta(z)}=\frac{f(z)}{\Delta(z)}=F(z)$

We now rewrite our functions in terms of the Ramanujan τ-function.

$$\Delta(z)=\sum_{n=1}^{\infty}\tau(n)q^n$$

$$f(z)=p^{12}\sum_{n=1}^{\infty}\tau(n)q^{pn}+\sum_{k=0}^{p-1}\sum_{n=1}^{\infty}\tau(n)q^{\frac{n}{p}}e^{\frac{2\pi ikn}{p}}$$

$$f(z)=p^{12}\sum_{n=1}^{\infty}\tau(n)q^{pn}+\sum_{n=1}^{\infty}\tau(n)q^{\frac{n}{p}}\sum_{k=0}^{p-1}e^{\frac{2\pi ikn}{p}}$$

Let

$$\varphi(p,n)=\sum_{k=0}^{p-1}e^{\frac{2\pi ikn}{p}}$$

If $p\mid n$, then $e^{2\pi ik\frac{n}{p}}=e^{2\pi im}$ where $m\in\mathbb{Z}$. It follows that $e^{2\pi ik\frac{n}{p}}=1\Rightarrow\varphi(p,n)=p$
Otherwise, we know that the sum of the pth roots of unity is 0, namely $\varphi(p,n)=0$

It follows that

$$f(z) = p^{12}\sum_{n=1}^{\infty}\tau(n)q^{pn} + p\sum_{l=1}^{\infty}\tau(lp)q^{\frac{lp}{p}} = p^{12}\sum_{n=1}^{\infty}\tau(n)q^{pn} + p\sum_{n=1}^{\infty}\tau(pn)q^{n}$$

Since $f(z) = \Delta(z)F(z)$, the degree one coefficient of $f(z)$ must correspond to the constant term of $F(z)$ since $\Delta(z)$ is a cusp from. Furthermore since $F(z) = F(z+1)$ and $F(-\frac{1}{z}) = F(z)$ and since the transformations $z \mapsto z+1$ and $z \mapsto -\frac{1}{z}$ generate the entire modular group, we can conclude that $F(z)$ is a weakly modular function of weight 0, and therefore must be constant. So,

$$F(z) = p\tau(p) \tag{10}$$

This gives us the identity

$$\tau(p)\sum_{n=1}^{\infty}\tau(n)q^{n} = p^{11}\sum_{n=1}^{\infty}\tau(n)q^{pn} + \sum_{n=1}^{\infty}\tau(pn)q^{n} \tag{11}$$

Take the nth coefficient of q in the expression above and we get:

$$\tau(p)\tau(p^{k-1}n) = p^{11}\tau(p^{k-2}n) + \tau(p^{k}n)$$

taking $n = 1$ gives us the desired result.

□

We can now proceed with the rest of the theorem.

Proving the multiplicative property equates to proving the expression:

$$\tau(p^{k}n) = \tau(p^{k})\tau(n) \tag{12}$$

If $k = 0$ then 12 holds trivially. Furthermore since $p \geq 2$, equating the n-th coefficient equates to

$$\tau(pn) = \tau(p)\tau(n) \tag{13}$$

Equating coefficients of (11)

To finish conclude the proof, we construct a recursive sequence on $k \in \mathbb{Z}^{+}$ given by:

$$\mu_k = \tau(p^{k}n) - \tau(p^{k})\tau(n) \tag{14}$$

As shown above $\mu_0 = \mu_1 = 0$, so suppose that $\mu_l = 0$ for all $0 \leq l \leq k-1$

Using 11 we derive the following auxiliary equations which give us (6) for $n = 1$

1. $\tau(p)\tau(p^{k-1}n) = p^{11}\tau(p^{k-2}n) + \tau(p^{k}n)$
2. $\tau(p)\tau(p^{k-1}) = p^{11}\tau(p^{k-2}) + \tau(p^{k})$

It follows that

$$\mu_{k-1} = \tau(p^{k-1}n) - \tau(p^{k-1})\tau(n) = 0$$
$$\mu_{k-2} = \tau(p^{k-2}n) - \tau(p^{k-2})\tau(n) = 0$$

Proof. Some simple algebraic manipulation shows that

$$\tau(p^k n) = \mu_k - \tau(p^k)\tau(n) = \tau(p)\tau(p^{k-1}n) - p^{11}\tau(p^{k-2}n)$$

$$\iff \mu_k = \tau(p)\tau(p^{k-1}n) - p^{11}\tau(p^{k-2}n) + \tau(p^k)\tau(n)$$

$$\iff \mu_k = [\tau(p)\tau(p^{k-1}n) - p^{11}\tau(p^{k-2}n)] - \tau(n)[\tau(p)\tau(p^{k-1}) - p^{11}\tau(p^{k-2})]$$

$$\iff \mu_k = \tau(p)[\tau(p^{k-1}n) - \tau(p^{k-1})\tau(n)] + p^{11}[\tau(n)\tau(p^{k-2}) - \tau(p^{k-2}n))]$$

$$\iff \mu_k = \tau(p)\mu_{k-1} - p^{11}\mu_{k-2}$$

$$\iff \mu_k = \tau(p^k n) - \tau(p^k)\tau(n) = 0$$

which proves that for any $k \in \mathbb{Z}^+$

$$\tau(p^k n) = \tau(p^k)\tau(n)$$

Thus for any relatively prime $m, n \in \mathbb{Z}$, we have

$$\tau(mn) = \tau(m)\tau(n)$$

□

4 Original Work

We begin by generalizing the result of the multiplicative property with a simple corollary.

Corollary 6. *For any $n \in \mathbb{N}$ such that $n = p_1^{e_1}p_2^{e_2}...p_k^{e_k}$ is the non-trivial prime decomposition of n*

$$\tau(n) = \tau(p_1^{e_1}p_2^{e_2}...p_k^{e_k}) = \prod_{j=1}^{k} \tau(p_j^{e_j}) \tag{15}$$

Proof. Clearly

$$\tau(n) = \tau(p_1^{e_1}p_2^{e_2}...p_k^{e_k})$$

Now without loss of generality, let $n_1 = p_2^{e_2}...p_k^{e_k}$ so by (2)

$$\tau(n) = \tau(p_1^{e_1}n_1) = \tau(p_1^{e_1})\tau(n_1)$$

And hence

$$\tau(n) = \tau(p_1^{e_1}p_2^{e_2}...p_k^{e_k}) = \prod_{j=1}^{k} \tau(p_j^{e_j})$$

□

Though the multiplicative properties of the τ function does not commonly appear in this form, the primary focus tends to be on prime coefficients, as their values predict those of larger coefficients.

4.1 Structure of $\tau(p^n)$

Given our proof of the multiplicative property for relatively prime integers, we want to explore the structure of $\tau(p^k)$ for

Using the auxiliary equation:

$$\tau(p^k) = \tau(p)\tau(p^{k-1}) - p^{11}\tau(p^{k-2}) \tag{16}$$

We compute values of for $k \geq 2$, write $\Lambda = \tau(p)$ and $q = p^{11}$

$$\tau(p^2) = (\tau(p))^2 - q = \Lambda^2 - q$$

$$\tau(p^3) = \tau(p)\tau(p^2) - q\tau(p) = \Lambda\tau(p^2) - q\Lambda = \Lambda^3 - 2q\Lambda$$

$$\tau(p^4) = \tau(p)\tau(p^3) - q\tau(p^2) = \Lambda(\Lambda^3 - 2q\Lambda) - q(\Lambda^2 - q) = \Lambda^4 - 3q\Lambda^2 + q^2$$

From now on we write $\zeta_k = \tau(p^k)$. From (16) we have the recurrence relation for $k \geq 3$. Note $\zeta_0 = \tau(1) = 1$

$$\zeta_k = \Lambda\zeta_{k-1} - q\zeta_{k-2} \tag{17}$$

We use the following code to generate ζ_k for the first few k.

```
from sympy import symbols, collect

x, q = symbols('x q')
P = [0] * 26

P[1] = x
P[2] = x**2 - q

for k in range(3, 26):
    P[k] = collect(x * P[k - 1] - q * P[k - 2], x)

for k in range(1, 26):
    print(f"P_{k} =", P[k])
~
```

Which gives rise to the following polynomials

$$\zeta_0 = 1$$

$$\zeta_1 = \Lambda$$

$$\zeta_2 = -q + \Lambda^2$$

$$\zeta_3 = -2q\Lambda + \Lambda^3$$

$$\zeta_4 = q^2 + -3q\Lambda^2 + \Lambda^4$$

$$\zeta_5 = 3q^2\Lambda + -4q\Lambda^3 + \Lambda^5$$

$$\zeta_6 = -q^3 + 6q^2\Lambda^2 - 5q\Lambda^4 + \Lambda^6$$

$$\zeta_7 = -4q^3 + 10q^2\Lambda^3 - 6q\Lambda^5 + \Lambda^7$$
$$\zeta_8 = q^4 - 10q^3\Lambda^2 + 15q^2\Lambda^4 - 7q\Lambda^6 + \Lambda^8$$
$$\zeta_9 = 5q^4\Lambda - 20q^3\Lambda^3 + 21q^2\Lambda^5 - 8q\Lambda^7 + \Lambda^9$$
$$\zeta_{10} = -q^5 + 15q^4\Lambda^2 - 35q^3\Lambda^4 + 28q^2\Lambda^6 - 9q\Lambda^8 + \Lambda^{10}$$
$$\zeta_{11} = -6q^5\Lambda + 35q^4\Lambda^3 - 56q^3\Lambda^5 + 36q^2\Lambda^7 - 10q\Lambda^9 + \Lambda^{11}$$
$$\zeta_{12} = q^6 - 21q^5\Lambda^2 + 70q^4\Lambda^4 - 84q^3\Lambda^6 + 45q^2\Lambda^8 - 11q\Lambda^{10} + \Lambda^{12}$$
$$\zeta_{13} = 7q^6\Lambda - 56q^5\Lambda^3 + 126q^4\Lambda^5 - 120q^3\Lambda^7 + 55q^2\Lambda^9 - 12q\Lambda^{11} + \Lambda^{13}$$
$$\zeta_{14} = -q^7 + 28q^6\Lambda^2 - 126q^5\Lambda^4 + 210q^4\Lambda^6 - 165q^3\Lambda^8 + 66q^2\Lambda^{10} - 13q\Lambda^{12} + \Lambda^{14}$$

and so on and so forth...

We can see that by highlighting the entries as we have above, a skew pattern of Pascal's triangle emerges. Furthermore, if we rewrite the coefficients as combinations in a specific manner, a pattern begins to emerge with respect to the structure of $\tau(p^k)$ in terms of $\tau(p)$. This leads us to conjecture and prove the following result.

Lemma 7. *For all $n \geq 2$, p prime.*

$$\tau(p^n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k} \tag{18}$$

Proof. We proceed by induction. We know that for any prime p and for all $m \geq 2$

$$\tau(p^m) = \tau(p)\tau(p^{m-1}) - p^{11}\tau(p^{m-2})$$

So for $n = 2$

$$\tau(p^2) = \sum_{k=0}^{1} \binom{2-k}{k} (-p^{11})^k (\tau(p))^{2-2k} = (\tau(p))^2 - p^{11}$$

so the equation holds for $m = 2$ Assume the induction hypothesis holds for all $m \leq n$. Then

$$\tau(p^{n+1}) =$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-(k+1)}{k} (-p^{11})^{k+1} (\tau(p))^{n-2(k+1)+1}$$

Case: $n \equiv 0 \pmod 2$

$$n \equiv 0 \pmod 2 \Rightarrow \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor + 1$$

So $\tau(p^{n+1})$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-(k+1)}{k} (-p^{11})^{k+1} (\tau(p))^{n-2(k+1)+1}$$

$$= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k-1} (-p^{11})^k (\tau(p))^{n-2k+1}$$

$$= (\tau(p))^{n+1} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k-1} (-p^{11})^k (\tau(p))^{n-2k+1}$$

$$= \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{(n+1)-k}{k} (-p^{11})^k (\tau(p))^{(n+1)-2k}$$

So the equation holds for even n

Case: $n \equiv 1 \pmod 2$

$$n \equiv 1 \pmod 2 \Rightarrow \lfloor\frac{n}{2}\rfloor = \lfloor\frac{n-1}{2}\rfloor = \lfloor\frac{n+1}{2}\rfloor - 1$$

So $\tau(p^{n+1})$

$$= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-(k+1)}{k} (-p^{11})^{k+1} (\tau(p))^{n-2(k+1)+1}$$

$$= (\tau(p))^{n+1} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor} \binom{n-k}{k-1} (-p^{11})^k (\tau(p))^{n-2k+1}$$

$$= (\tau(p))^{n+1} + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k+1}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + (-p^{11})^{\frac{n+1}{2}}$$

$$= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k+1}{k} (-p^{11})^k (\tau(p))^{n-2k+1} + (-p^{11})^{\frac{n+1}{2}}$$

$$= \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{(n+1)-k}{k} (-p^{11})^k (\tau(p))^{(n+1)-2k}$$

So the equation holds for odd n and hence for all $n \geq 2$, which concludes the proof. □

Theorem 8. *For any $n \in \mathbb{N}$ such that $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$*

$$\tau(n) = \prod_{j=1}^{k} \sum_{l=0}^{\lfloor\frac{e_j}{2}\rfloor} \binom{e_j - l}{l} (-p_j^{11})^l (\tau(p_j))^{e_j - 2l} \tag{19}$$

Proof. The result follows immediately from (6) and (7) □

The significance of this result is that it formalizes the multiplicative properties of the Ramanujan tau function on its entire domain, given we know the value of prime coefficients. It also extends the understood notion of the "predictive power" of the prime coefficients.

5 Further Discussion of the τ function

Congruences The congruences of the tau function play a key role in understanding the number theoretic properties of Modular forms. It turns out that the space of all cusp forms of weight k denoted S_k is generated entirely by two Eisenstein series, which can be generalized and broken down using Lipschitz's formula. [6] Taking the Fourier expansion of the Eisenstein series of weight k one sees that

$$G_k(z) = \frac{1}{2} \sum_{0 \neq m,n \in \mathbb{Z}} \frac{1}{(mz+n)^k} = \zeta(k) + \frac{2\pi i}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $q = e^{2\pi i z}$, ζ is the Riemann-Zeta function and $\sigma_{k-1}(n)$ is the $k-1$'th power divisor function. G_k is a modular form for even $k \geq 3$, and there are no non-trivial modular forms of odd weight. By writing $\Delta(z)$ in terms of Eisenstein series, one can postulate that there may be a relationship between σ_m for some integers m and values of $\tau(n)$. One particular example that was noticed by Ramanujan[1]

$$\tau(n) \equiv \sigma_{11}(n) \mod 691$$

This can be seen by manipulating the modular forms E_4 and E_6 where

$$E_k(z) = \frac{1}{\zeta(k)} G_k(z) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

and B_k is the k-th Bernoulli number. We have the equations

$$E_{12}(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \text{ and } E_4^3 - E_6^2 = 1728\Delta$$

which gives us the congruence. (Note, 691 is prime) [3]

Riemann Hypothesis Another consequence of $\tau(n)$'s multiplicative properties is its associated L-function: [1]

$$L_\tau(s) = \sum_{n=0}^{\infty} \frac{\tau(n)}{n^s}$$

which has Euler product:

$$L_\tau(s) = \prod_{p \text{ - prime}} \frac{1}{1 - p^{-s}\tau(p) - p^{11-2s}}$$

L-functions are a class of analytic functions on the complex plane that have a similar property to the Riemann-Zeta hypothesis. Namely, that the function behaves similarly on either side of a "symmetry line" [5] that is hidden in the definition of the function. Much like the Riemann Hypothesis, Ramanujan conjectured that all zeros of $L_\tau(s)$ have real part equal to 6. [1] While our understanding of L-functions is not sufficient to grant us answers to the Riemann Hypothesis yet. It is believed that expanding our understanding of them by studying functions like L_τ can only help in our pursuit of a proof for the Generalized Riemann Hypothesis which if true, will revolutionize our understanding of prime numbers.[5]

In researching the τ function, it appears that it has also played a major role in our understanding of Elptic Curves, Representation theory, and it set the groundwork for some of the results in respresentation theory that led Andrew Wiles to his famous proof of Fermat's Last theorem in 1995. [8] [5]

6 Conclusion

As discussed in the paper, Ramanujan's discovery of the Tau function led to revolutionary work in Mathematics. From the perspective of understanding the structure of the τ function, the significance of creating a tangible representation of prime power expansions is that it creates an alternate way for us to construct values of the output, specifically how irreducible forms of the output interact with each other and build constructively to reveal an overarching structure. In developing such a formula, many of the famous results about the Tau function became more accessible and their link seemed to become marginally clearer with the representation. Given the impact that Ramanujan's work has already had on two of the largest problems in mathematics, Fermat's Last Theorem and the Riemann Hypothesis, it suffices to say that expanding our understanding the structure of the function will have a lasting impact on the field of mathematics as a whole.

References

[1] Srinivasa Ramanujan. "On Certain Arithmetical Functions". In: *Transactions of the Cambridge Philosophical Society* 22 (1916), pp. 169–184.

[2] Louis J. Mordell. "Mr. Mordell, on Mr. Ramanujan's Empirical Expansions of Modular Forms". In: *Proceedings of the Cambridge Philosophical Society* 19 (1917), pp. 117–124.

[3] Matt E. *Why is $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$?* Mathematics Stack Exchange. 2010. URL: https://math.stackexchange.com/q/15740.

[4] Robert Schneider. *Encounter With the Infinite*. The Believer. 2015. URL: https://www.thebeliever.net/encounter-with-the-infinite/.

[5] Brady Haran and John Keating. *The Key to the Riemann Hypothesis*. YouTube video. 2016. URL: https://youtu.be/VTveQ1ndH1c.

[6] Henrik Bachmann. "Introduction to Modular Forms". In: *Perspectives in Mathematical Science* 4 (2018), pp. 1–26.

[7] Paramanand Singh. *Some Convincing Reasoning to Show That to Prove That Ramanujan Tau Function is Multiplicative Is Very Difficult*. Mathematics Stack Exchange. 2018. URL: https://math.stackexchange.com/q/2666087.

[8] Alex Contorovich. *What is the Langlands Program*. Quanta Magazine. 2022. URL: https://www.quantamagazine.org/what-is-the-langlands-program-20220601.